21-301 Combinatorics
Homework 9
Due: Monday, November 23

1. Consider the following take-away game: There is a pile of n chips. A move consists of removing 3^k chips for some $k \geq 1$. Compute the Sprague-Grundy numbers $g(n)$ for $n \geq 0$.

   Solution: After looking at the first few numbers $0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, \ldots$ one sees that

   $$g(n) = \begin{cases} 0 & n = 0, 1, 2 \mod 6 \\ 1 & n = 3, 4, 5 \mod 6 \end{cases}$$

   We verify this by induction. It is true for $n \leq 6$ by inspection. For $n > 6$ we have $g(n) = \text{mex}\{g(n-3), g(n-9), \ldots, \}$. Observe that if $k \geq 2$ then $3^k = 3(3^{k-1} - 1) + 3$ and so $3^k \mod 6 = 3$. It follows that $g(n) = \text{mex}\{g(n-3)\}$ and the induction step follows.

2. **Poker Nim:** In this game there is a collection of piles of chips plus an extra bag containing a finite number of chips. For a move one can either (i) make a regular Nim move or (ii) take some chips from the bag and put them onto one of the piles. How should one play this game?

   Solution: To play the game one evaluates the position as in Nim. A position is N/P iff it is N/P in Nim without the bag. If a player X is in a losing position and takes p chips from the bag and places it on pile h then the next player can respond by removing p chips from pile h. Thus player X will still be in a losing Nim position. Eventually the bag becomes empty and we are back to standard Nim.

3. Consider the following game: There is a pile of chips. A move consists of removing s chips where $s \in S$, assuming that there are at least s chips left. If $|S|$ is finite, show that the Sprague-Grundy numbers satisfy $g(n) \leq |S|$ where n is the number of chips remaining.

   Solution: Observe that for any finite set A, $mex(A) \leq |A|$ since $mex(A) > |A|$ implies that $A \subseteq \{0, 1, 2, \ldots, |A|\}$ which is obviously impossible. In the take-away game $g(n)$ is the mex of a set of size at most $|S|$ and the result follows.