21-301 Combinatorics
Homework 8
Due: Monday, November 16

1. Let $\mathcal{A}$ be an intersecting family of subsets of $[n]$ such that $A \in \mathcal{A}$ implies $k \leq |A| \leq \ell \leq n/2$. Show that
$$|\mathcal{A}| \leq \sum_{i=k}^{\ell} \binom{n-1}{i-1}.$$
Solution: Let $\mathcal{A}_i = \{A \in \mathcal{A} : |A| = i\}$. Then $\mathcal{A}_i$ is an intersecting family and so by the Erdős-Ko-Rado theorem, we have $|\mathcal{A}_i| \leq \binom{n-1}{i-1}$ and the result follows from $|\mathcal{A}| = |\mathcal{A}_k| + \cdots + |\mathcal{A}_\ell|$.

2. Let $m = \lfloor n/2 \rfloor$. Describe a family $\mathcal{A}$ of size $2^{n-1} + \binom{n-1}{m-1}$ that has the following property: If $A_1, A_2 \in \mathcal{A}$ are disjoint then $A_1 \cup A_2 = [n]$.
Solution: If $n = 2m+1$ is odd, let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ where $\mathcal{A}_1 = \{A \subseteq [n] : \ |A| \geq m+1\}$ and $\mathcal{A}_2 = \{A \subseteq [n] : \ |A| = m, \ A \ni 1\}$. Here $|\mathcal{A}_1| = 2^{n-1}$, because if we partition the subsets of $[n]$ into 2^{n-1} pairs, a set and its complement, then the larger of the two sets is in $\mathcal{A}_1$. Clearly $|\mathcal{A}_2| = \binom{n-1}{m-1}$. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ are intersecting families and if $A \in \mathcal{A}_1, B \in \mathcal{A}_2$ and $A \cap B = \emptyset$ then we have $|B| = m$ and $|A| \geq m+1$ and so A, B must be complementary.
If $n = 2m$ is even then let $\mathcal{A} = \{A \subseteq [n] : \ |A| \geq m\}$. Now
$$|\mathcal{A}| = |\mathcal{A}_1| + \binom{n}{m} = |\mathcal{A}_1| + \frac{1}{2}\binom{n}{m} + \frac{1}{2}\binom{n}{m} = 2^{n-1} + \frac{1}{2}\binom{n}{m} = 2^{n-1} + \binom{n-1}{m-1}.$$
Here, $|\mathcal{A}_1| + \frac{1}{2}\binom{n}{m} = 2^{n-1}$ because we can obtain this number of sets by taking one set from each pair of complementary sets. Each $A \in \mathcal{A}_1$ intersects all sets of size m or more and two sets of size m fail to intersect only when they are complementary.

3. Consider the following game: There is a pile of n chips. A move consists of removing any *proper* factor of n chips from the pile. (For the purposes of this question, a proper factor of n, is any factor, including 1, that is strictly less than n). The player to leave a pile with one chip wins. Determine the N and P positions and a winning strategy from an N position.
Solution: n is a P-position iff it is odd. If n is even then the next player can simply remove one chip. If n is odd, then any factor of n is also odd.