21-301 Combinatorics
Homework 7
Due: Friday, November 6

1. Use the pigeon-hole principle to show that for every integer $k \geq 1$ there exists a power of 3 that ends with $000 \cdots 0001$ (k 0's).

   Solution: If we consider the infinite sequence $u_\ell = 3^\ell \mod 10^{k+1}$ for $\ell = 1, 2, \ldots,$ then by the PHP there exist $m < n$ such that $u_m = u_n$, In which case,

   $$3^n - 3^m = 10^{k+1}s \text{ or } 3^{n-m}(3^m - 1) = 10^{k+1}s$$

   for some positive integer s.

   Now 3 and 10 are co-prime and therefore $3^m - 1 = 10^{k+1}s'$ for some positive integer s', and this implies the result.

2. Show that if the edges of K_{m+n} are colored red and blue then either (i) there is a red path with m edges or (ii) a vertex of blue degree at least n.

   Solution: If there is no vertex of blue degree at least n then the red graph has minimum degree at least m. Let $P = x_1, x_2, \ldots, x_k$ be a longest path in the red graph. All of x_k's neighbors in the red graph lie on P, else P can be extended. But x_k has at least m neighbours and so $k \geq m + 1$.

3. Show that if $n = 2m$ is even and the edges of K_n are colored red or blue then either (i) there is a red triangle or (ii) there is a vertex of blue degree at least $m - 1$.

   Solution: If the blue graph has maximum degree $m - 2$ then the blue graph contains an independent set of size $\frac{n^2}{n(m-2)+n} > 2$ i.e. it contains an idependent set of size three and this corresponds to a red triangle.