# 21-301 Combinatorics
# Homework 6
# Due: Monday, November 2

1. Let $\chi(G)$ be the chromatic number of graph $G = (V, E)$. Let $\alpha(G), \kappa(G)$ denote the size of the largest independent set of $G$, clique of $G$ respectively.

   Show that
   $$\chi(G) \geq \max\left\{\frac{|V|}{\alpha(G)}, \kappa(G)\right\}.$$

   Show further that $\chi(G)\chi(\bar{G}) \geq |V|$.

   Here $\bar{G}$ is the complement of $G$ i.e. the graph with edge set $\binom{V}{2} \setminus E$.

   **Solution:** If $S$ is a clique of size $s$ then we need at least $s$ colors, just to color $S$. Thus $\chi(G) \geq \kappa(G)$. Observe next that in a proper coloring, a set of vertices of the same color must form an independent set. Thus a proper coloring partitions the vertex set into a set of color classes $C_1, C_2, \ldots, C_k$ where $|C_i| \leq \alpha(G)$ for $1 \leq i \leq k$. This implies that $k \geq |V|/\alpha(G)$.

   For the second part, we have
   $$\chi(G) \geq \frac{|V|}{\alpha(G)} = \frac{|V|}{\kappa(\bar{G})}.$$

   Hence,
   $$\chi(G)\chi(\bar{G}) \geq \chi(G)\kappa(\bar{G}) \geq \frac{|V|}{\kappa(\bar{G})}\kappa(\bar{G}) = |V|.$$

2. Let $G = (V, E)$ be a graph with $kn$ vertices. Show, by the probabilistic method, that there is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ with $|V_i| = n$, $i = 1, 2, \ldots, k$ such that at most $|E|/k$ of the edges of $G$ have both of their endpoints in the same part of the partition.

   **Solution:** Let $V_1, V_2, \ldots, V_k$ be a random partition of the vertex set. Let $e = (v, w)$ be an edge of $E$. Then
   $$\Pr(\exists i :\ e \subseteq V_i) = \sum_{i=1}^{k} \Pr(w \in V_i \mid v \in V_i) \Pr(v \in V_i) =$$
   $$\sum_{i=1}^{k} \frac{\binom{kn-2}{n-2}}{\binom{kn-1}{n-1}} \frac{1}{k} = \frac{n-1}{kn-1} < \frac{1}{k}.$$

   If $Z$ is the number of edges of $G$ that have both of their endpoints in the same part of the partition, then $\mathbf{E}(Z) \leq |E|/k$ and so the required partition must exist.

3. Let $P_1, P_2$ be two paths of maximum length in a connected graph $G$. Prove that $P_1, P_2$ share a common vertex.

   **Solution** Suppose that $P_1, P_2$ are vertex disjoint. Let $P_1 = (u_1, u_2, \ldots, u_k)$ and $P_2 = (v_1, v_2, \ldots, v_k)$. Let $Q = (u_i = w_1, w_2, \ldots, w_\ell = v_j)$ be a shortest path from a vetex in $P_1$ to a vertex in $P_2$. Such a path exists because $G$ is connected. Now $w_2, w_3, \ldots, w_{\ell-1}$ are not vertices of $P_1$ or $P_2$, since $Q$ is a shortest path. Assume w.l.o.g. that $i, j \geq k/2$. Then the path $u_1, u_2, \ldots, u_i, w_2, \ldots, w_{\ell-1}, u_j, u_{j-1}, \ldots, u_1$ has length at least $i+j+1 \geq k+1$, contradiction.