21-301 Combinatorics
Homework 5
Due: Monday, October 26

1. Prove that if u, v are the only vertices of odd degree in a graph G, then there is a path from u to v in G.

   Solution: We have to show that u, v are in the same component of G. But is they are in different components, $u \in C_1, v \in C_2$ then the sub-graph induced by C_1 has one odd vertex, u. This contradicts the fact that every graph has an even number of vertices.

2. Let $G = (V, E)$ be a graph with minimum degree at least three. Show that it contains a cycle of even length. (Hint: Consider a longest path).

   Solution: Let $P = (x = x_0, x_1, \ldots, x_k)$ be a longest path in G. Let $x_1, x_i, x_j, 1 < i < j$ be three neighbors of x. If i is odd then the cycle $(x_0, x_1, \ldots, x_i, x_0)$ has $i + 1$ edges and is even and so we can assume that i, j are both even. But then the cycle $(x_0, x_i, x_{i+1}, \ldots, x_j, x_0)$ has $j - i + 2$ edges and is even.

3. Prove that if $T_1, T_2, \ldots, T_k$ are pair-wise intersecting sub-trees of a tree T, then T has a vertex common to $T_1, T_2, \ldots, T_k$. (Hint: use induction on k).

   Solution: Assume inductively that $H = \bigcap_{i=1}^{k} T_i$ is non-empty. H must be a sub-tree of T, for if $u, v \in H$ then each T_i contains the path from u to v in T. Now let $\Gamma = T \setminus H$ be obtained by deleting the vertices of H from T. Let $C_1, C_2, \ldots, C_m$ be the components of Γ. Each C_i contains a unique vertex v_i that is adjacent to Γ. If C_1 contained two such vertices u, u' then either the path from u to u' goes through Γ and then u, u' are in different components of Γ or it avoids Γ and then T contains a cycle, contradiction. Suppose now that T_{k+1} does not share a vertex with Γ. Then T_{k+1} must be contained in a single component C_1, say. For if T_{k+1} meets C_1 and C_2 then T_{k+1} must contain a path from C_1 to C_2 and this must go through Γ. We claim now that v_1 belongs to $T_1, T_2, \ldots, T_{k+1}$. Suppose that $w \in C_1$ is in T_1 and T_{k+1}. Then T_1 contains a path from w to Γ and this goes through v_1. But then $v_1 \in \Gamma$, contradiction.