21-301 Combinatorics
Homework 3
Due: Monday, September 28

1. Suppose that in the Tower of Hanoi problem there are n sets of k rings of the same size. For example you there could be two rings of size 1, two rings of size 2 and 2 rings of size 3, here $n = 3$ and $k = 2$. You can put a ring onto another ring of the same size or larger. How long does it take to move the rings on Peg 1 to peg 3 under these circumstances?

   Solution: Let $H_{n,k}$ be the minimum number of moves required. Then

   $$H_{n,k} = 2H_{n-1,k} + k$$

   for $n \geq 2$. $H_{1,k} = k$.

   One can solve this eqaution using generating functions or just notice that

   $$H_{n,k} = kH_{n,1} = k(2^n - 1).$$

   This is true for $n = 1$ and inductively

   $$H_{n,k} = 2H_{n-1,k} + k = k(2H_{n-1,1} + 1) = kH_{n,1}.$$

2. Show that the number of sequences out of $\{a, b, c\}^n$ which do not contain a consecutive sub-sequence of the form abc satisfies the recurrence $b_0 = 1, b_1 = 3, b_2 = 9$ and

   $$b_n = 2b_{n-1} + c_n \qquad (1)$$
   $$c_n = c_{n-1} + b_{n-2} + c_{n-2} + b_{n-3} \qquad (2)$$

   where c_n is the number of such sequences that start with a.

   Now find a recurrence only involving b_n, by using (1) to eliminate c_n from (2).

   Solution: There are $2b_{n-1}$ sequences of the required form that start with b or c. There are c_n sequences that start with a. This explains (1).

   There are c_{n-1} sequences that start with aa, b_{n-2} sequences that start with ac, c_{n-2} sequences that start with aba and b_{n-3} sequences that start with abb. This covers the possibilities for sequences starting with a.

   We have

   $$b_n - 2b_{n-1} = b_{n-1} - 2b_{n-2} + b_{n-2} + b_{n-2} - 2b_{n-3} + b_{n-3}$$

   and so

   $$b_n = 3b_{n-1} - b_{n-3}.$$

3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the recurrence relation

   $$a_n + 3a_{n-1} + 2a_{n-2} = n + 1 \quad \text{for } n \geq 2$$

   with initial conditions $a_0 = 1$ and $a_1 = 3$. Determine the generating function for this sequence, and use the generating function to determine a_n for all n.

Solution:

$$\begin{aligned}
\sum_{n=2}^{\infty}(a_n + 3a_{n-1} + 2a_{n-2})x^n &= \sum_{n=2}^{\infty}(n+1)x^n \\
a(x) - 1 - 3x + 3x(a(x) - 1) + 2x^2 a(x) &= \frac{1}{(1-x)^2} - 1 - 2x \\
a(x)(1 + 3x + 2x^2) &= \frac{1}{(1-x)^2} + 4x
\end{aligned}$$

$$\begin{aligned}
a(x) &= \frac{1}{(1+x)(1+2x)(1-x)^2} + \frac{4x}{(1+x)(1+2x)} \\
&= \frac{15/4}{1+x} + \frac{-28/9}{1+2x} + \frac{7/36}{1-x} + \frac{1/6}{(1-x)^2} \\
&= \sum_{n=0}^{\infty}\left(\frac{15}{4}(-1)^n - \frac{28}{9}(-2)^n + \frac{7}{36} + \frac{1}{6}(n+1)\right)x^n.
\end{aligned}$$

So

$$a_n = \frac{15}{4}(-1)^n - \frac{28}{9}(-2)^n + \frac{7}{36} + \frac{1}{6}(n+1) \qquad \text{for } n \geq 0.$$