21-301 Combinatorics
Homework 3
Due: Monday, September 28

1. Suppose that in the Tower of Hanoi problem there are n sets of k rings of the same size. For example you there could be two rings of size 1, two rings of size 2 and 2 rings of size 3, here $n = 3$ and $k = 2$. You can put a ring onto another ring of the same size or larger. How long does it take to move the rings on Peg 1 to peg 3 under these circumstances?

2. Show that the number of sequences out of $\{a, b, c\}^n$ which do not contain a consecutive sub-sequence of the form abc satisfies the recurrence $b_0 = 1, b_1 = 3, b_2 = 9$ and

$$b_n = 2b_{n-1} + c_n \quad (1)$$
$$c_n = c_{n-1} + b_{n-2} + c_{n-2} + b_{n-3} \quad (2)$$

   where c_n is the number of such sequences that start with a.

   Now find a recurrence only involving b_n, by using (1) to eliminate c_n from (2).

3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the recurrence relation

$$a_n + 3a_{n-1} + 2a_{n-2} = n + 1 \quad \text{for } n \geq 2$$

   with initial conditions $a_0 = 1$ and $a_1 = 3$. Determine the generating function for this sequence, and use the generating function to determine a_n for all n.