21-301 Combinatorics
Homework 2
Due: Friday, September 18

1. Prove that for any $k, n \geq 1$ that

$$\sum_{\substack{a_1+\cdots+a_{2k}=n \\ a_1,\ldots,a_{2k}\geq 0}} (-1)^{a_1+\cdots+a_k} \binom{n}{a_1,\ldots,a_{2k}} = 0.$$

Solution: We start with the expression

$$(x_1 + x_2 + \cdots + x_{2k})^n = \sum_{\substack{a_1+\cdots+a_{2k}=n \\ a_1,\ldots,a_{2k}\geq 0}} \binom{n}{a_1,\ldots,a_{2k}} x_1^{a_1} x_2^{a_2} \cdots x_{2k}^{a_{2k}}$$

and then put $x_1 = x_2 = \cdots = x_k = -1$ and $x_{k+1} = x_{k+2} = \cdots = x_{2k} = 1$ to get the result we want.

2. **(a)** Let $\mathcal{S}_k$ denote the collection of k-sets $\{1 \leq i_1 < i_2 < \cdots < i_k \leq m-3\} \subseteq [m]$ such that $i_{t+1} - i_t \geq 4$ for $1 \leq t < k$. Show that

$$|\mathcal{S}_k| = \binom{m-3k}{k}.$$

(b) How many of the 4^n sequences $x_1x_2\cdots x_n$, $x_i \in \{a,b,c,d\}$, $i = 1,2,\ldots,n$ are there such that *abcd* does not appear as a consecutive subsequence e.g. if $n = 6$ then we include *adbbcc* in the count, but we exclude *aabcda*.

[You should use Inclusion-Exclusion and expect to have your answer as a sum.]

Solution:

(a) For a first argument, let $z_1 = i_1, z_2 = i_2 - i_1, \ldots, z_k = i_k - i_{k+1}, z_{k+1} = m - i_k$. We can count the number of choices for $z_1, z_2, \ldots, z_{k+1}$. But these are the solutions to

$$z_1 + z_2 + \cdots + z_{k+1} = m,\ z_1 \geq 1, z_2, z_3, \ldots, z_k \geq 4, z_{k+1} \geq 3.$$

The number of such is

$$\binom{m-1-4(k-1)-3+k+1-1}{k+1-1} = \binom{m-3k}{k}.$$

Alternatively, we can represent a k-set by a sequence of k 1's and $m-k$ 0's in the usual way. Now we need every pair of 1's separated by at least 3 0's. We can start with a sequence of $m-3k$ 0's, choose k of them and replace each of these k 0's by 1000. This process is reversible. For the 0,1 sequences we are counting each 1 is followed by at least 3 0's. Just replace 1000 by 0 to get a sequence of $m-k$ 0's.

(b) Let $A = \{a,b,c,d\}^n$. Then let

$$A_k = \{x \in A :\ x_k = a, x_{k+1} = b, x_{k+2} = c, x_{k+3} = d\}$$

for $k = 1, 2, \ldots, n-3$.

Let $\mathcal{S} = \bigcup_{k \geq 0} \mathcal{S}_k$. Then

$$|A_S| = \begin{cases} 4^{n-4|S|} & S \in \mathcal{S}_{|S|} \\ 0 & S \notin \mathcal{S}_{|S|} \end{cases}.$$

Then we must compute

$$\begin{aligned} \left| \bigcap_{i=1}^{m} \bar{A}_i \right| &= \sum_{S \in \mathcal{S}} (-1)^{|S|} |A_S| \\ &= \sum_{S \in \mathcal{S}_{|S|}} (-1)^{|S|} 4^{n-4|S|} \\ &= 4^n \sum_{k=0}^{m} (-1)^k |\mathcal{S}_k| 4^{-4k} \\ &= 4^n \sum_{k=0}^{m} (-1)^k | \binom{m-3k}{k} 4^{-4k} \end{aligned}$$

3. How many ways are there of placing m distinguishable balls into n boxes so that no box contains more than $m/2$ balls.
   (You should use Inclusion-Exclusion and expect to have your answer as a sum.)

   Solution: Let A be the set of n^m allocations of balls to boxes. Let A_i denote the allocations in A in which box i gets at least $\mu = \lfloor m/2 \rfloor + 1$ balls. (μ is the smallest integer greater than $m/2$). We want the size of $\bigcap_{i=1}^{m} \bar{A}_i$.

   Now

$$|A_i| = \sum_{k=\mu}^{m} \binom{m}{k} (n-1)^{m-k}$$

   and $A_S = \emptyset$ for $|S| \geq 2$. So,

$$\left| \bigcap_{i=1}^{m} \bar{A}_i \right| = n^m - n \sum_{k=\mu}^{m} \binom{m}{k} (n-1)^{m-k}.$$