21-301 Combinatorics
Homework 1
Due: Friday, September 5

1. How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100$$

satisfy $x_1 \geq 4$, $x_2 \geq 8$, $x_3 \geq -2$, $x_4 \geq 3$ and $x_5 \geq 0$?

Solution Let

$$y_1 = x_1 - 4, \quad y_2 = x_2 - 8, \quad y_3 = x_3 + 2, \quad y_4 = x_4 - 3, \quad y_5 = x_5.$$

An integral solution of $x_1 + x_2 + x_3 + x_4 + x_5 = 100$ such that $x_1 \geq 4$, $x_2 \geq 8$, $x_3 \geq -2$, $x_4 \geq 3$ and $x_5 \geq 0$ corresponds to an integral solution of $y_1 + y_2 + y_3 + y_4 + y_5 = 87$ such that $y_1, \ldots, y_5 \geq 0$. From a result in class,

$$|\{(y_1, y_2, y_3, y_4, y_5) : y_1, \ldots, y_5 \in Z_+ \text{ and } y_1 + \cdots + y_5 = 87\}| = \binom{87+5-1}{5-1} = \binom{91}{4}.$$

2. Show that if $n \geq q \geq 0$ then

$$\sum_{k=0}^{\ell} \binom{\ell - k}{m} \binom{q+k}{n} = \binom{\ell + q + 1}{m+n+1}.$$

Solution Let $S = \binom{[\ell+q+1]}{m+n+1}$. If $\{x_1 < x_2 < \cdots < x_{m+n+1}\} \in S$ then put X in S_k if $x_{m+1} = \ell - k + 1$. Our assumption $n \geq q$ implies that $x_{m+1} \leq \ell + 1$ and so $0 \leq k \leq \ell$. The sets $S_0, S_1, \ldots, S_k$ partition S and $|S_k| = \binom{\ell-k}{m}\binom{q+k}{n}$.

3. How many ways are there of placing k 1's and $n-k$ 0's at the vertices of an n vertex polygon, so that every pair of 1's are separated by at least ℓ 0's?

Solution Choose a vertex v of the polygon in n ways and then place a 1 there. For the remainder we must choose $a_1, \ldots, a_k \geq \ell$ such that $a_1 + \cdots + a_k = n - k$ and then go round the cycle (clockwise) putting a_1 0's followed by a 1 and then a_2 0's followed by a 1 etc..

Each pattern π arises k times in this way. There are k choices of v that correspond to a 1 of the pattern. Having chosen v there is a unique choice of $a_1, a_2, \ldots, a_k$ that will now give π.

There are $\binom{n-k\ell-1}{k-1}$ ways of choosing the a_i and so the answer to our question is

$$\frac{n}{k}\binom{n-k\ell-1}{k-1}.$$