9\4\09

Proof of PIE

$$\left| \bigcap_{i=1}^{N} \overline{A}_i \right| = ??$$

$$\Theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

$$(1-\theta_{x,1})(1-\theta_{x,2})(1-\theta_{s})\cdots(1-\theta_{x,N})$$

$= 1$ iff all terms are 1

if $\theta_{x,1} = \theta_{x,2} = \cdots = \theta_{x,N} = 0$

if $x \in \bigcap_{i=1}^{N} \overline{A_i}$

So,

$$\left|\bigcap_{i=1}^{N} \bar{A}_i\right| = \sum_{x\in A}(1-\theta_{x,1})(1-\theta_{x,2})\cdots(1-\theta_{x,N})$$

$$= \sum_{x\in A}\sum_{S\subseteq[N]}(-1)^{|S|}\prod_{i\in S}\theta_{x,i}$$

(where I choose θ

$$= \sum_{S\subseteq[N]}(-1)^{|S|}\underbrace{\underbrace{\sum_{x\in A}\prod_{i\in S}\theta_{x,i}}_{}}_{|A_S|}$$

$=1$ iff $x\in\bigcap_{i\in S}A_i$

$= A_S$

Euler's Totient Function

$\phi(n)$ = number of $x \leq n$ such that h.c.f. $(x, n) = 1$

$\phi(18) = 6$

$18 = 2 \times 3^2 \quad \phi(18) = 18\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)$
$= 6$

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

= prime factorization

x & n are co-prime iff

$p_1 \nmid x$ & $p_2 \nmid x$ & ...

$x \notin A_1$ & $x \notin A_2$ & ...

$A_i := \{ y \leq n : p_i | y \}$

$$\phi(n) = \left| \bigcap_{i=1}^{k} \overline{A_k} \right|$$

$$= \sum_{S \subseteq [k]} (-1)^{|S|} |A_S|$$

$$A_i := \{ x \leq n : p_i \mid x \}$$

$$|A_i| = \frac{n}{p_i}$$

$$|A_{\{i,j\}}| = \frac{n}{p_i \, p_j}$$

$$\vdots$$

$$|A_S| = \frac{n}{\prod_{i \in S} p_i}$$

$$\phi(n) = \sum_{S \subseteq [k]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i}$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

<u>Euler</u>

Surjections:

\# functions $f:[n] \longrightarrow [m]$

that are onto.

f is a surjection if

$\bar{A}_1$ $\bar{A}_2$

$(\exists x \text{ s.t. } f(x)=1)$ & $(\exists x \text{ s.t. } f(x)=2)$ &

$(\exists x \text{ s.t. } f(x)=3)$ & $\ldots$ & $(\exists x : f(x)=m)$

$\bar{A}_3$ $\bar{A}_m$

$A_i = \{f : \nexists x \text{ s.t. } f(x)=i\}$

#surjections =

$$\sum_{S \subseteq [m]} (-1)^{|S|} |A_S|$$

$|A_S| = ?$

$|A_\emptyset| = |\text{all functions}| = m^n$

$|A_1| = |\{f : f(x) \neq 1, \forall x\}| = (m-1)^n$

$$|A_{\{1,2\}}| = |\{f : f(x) \neq 1, 2, \forall x\}|$$

$$= (m-2)^n$$

$$|A_S| = (m-|S|)^n$$

surjections

$$= \sum_{S \subseteq [m]} (-1)^{|S|} (m-|S|)^n = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

$= 0$ if $m > n$??