9\2\09

Principle of Inclusion – Exclusion

PIE

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$A_1, A_2 \subseteq A$ and $\bar{A}_1 = A \setminus A_1$

$$|\bar{A}_1 \cap \bar{A}_2| = |A| - |A_1| - |A_2| + |A_1 \cap A_2|$$

$\overline{A_1 \cup A_2}$ De Morgan's Rule.

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| =$$

$$|A|$$

$$- |A_1| - |A_2| - |A_3|$$

$$+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$$

$$- |A_1 \cap A_2 \cap A_3|$$

$$A_1, A_2, \ldots, A_N \subseteq A$$

$$\left| \bigcap_{i=1}^{N} \overline{A_i} \right| =$$

$$|A|$$

$$- |A_1| - |A_2| - \cdots - |A_N|$$

$$+ |A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{N-1} \cap A_N|$$

$$- |A_1 \cap A_2 \cap A_3| - |A_1 \cap A_2 \cap A_4| - \cdots - |A_{N-2} \cap A_{N-1} \cap A_N|$$

$$+ |A_1 \cap A_2 \cap A_3 \cap A_4| + \cdot$$

$S \subseteq [N]$ (nothing to do with A.)

$$A_S = \bigcap_{i \in S} A_i$$

$$= \{x \in A : x \in A_i, \forall i \in S\}$$

$$A_{\{3,5,12\}} = A_3 \cap A_5 \cap A_{12}$$

$$A_\emptyset = A$$

$$\left|\bigcap_{i=1}^{N} \overline{A}_i\right| =$$

$$\sum_{S \subseteq [n]} (-1)^{|S|} |A_S|$$

Simple Example

How many integers $\leq 10^6$ are not divisible by 5, 8, 12 ?

(Not divisible by 5) $A_1 = \{\text{divisible by } 5\}$

and

(Not divisible by 8) $A_2 = \{\text{divisible by } 8\}$

and

(Not divisible by 12) $A_3 = \{\text{divisible by } 12\}$

$\# =$

10^6 a

$- 200{,}000 - 125{,}000 - 50{,}000$ 1

$+ 25{,}000 + 16{,}666 + 41{,}666^{*}$ 2

$- 8{,}333$ 3

$* \left\lfloor \frac{10^6}{24} \right\rfloor$

Derangements

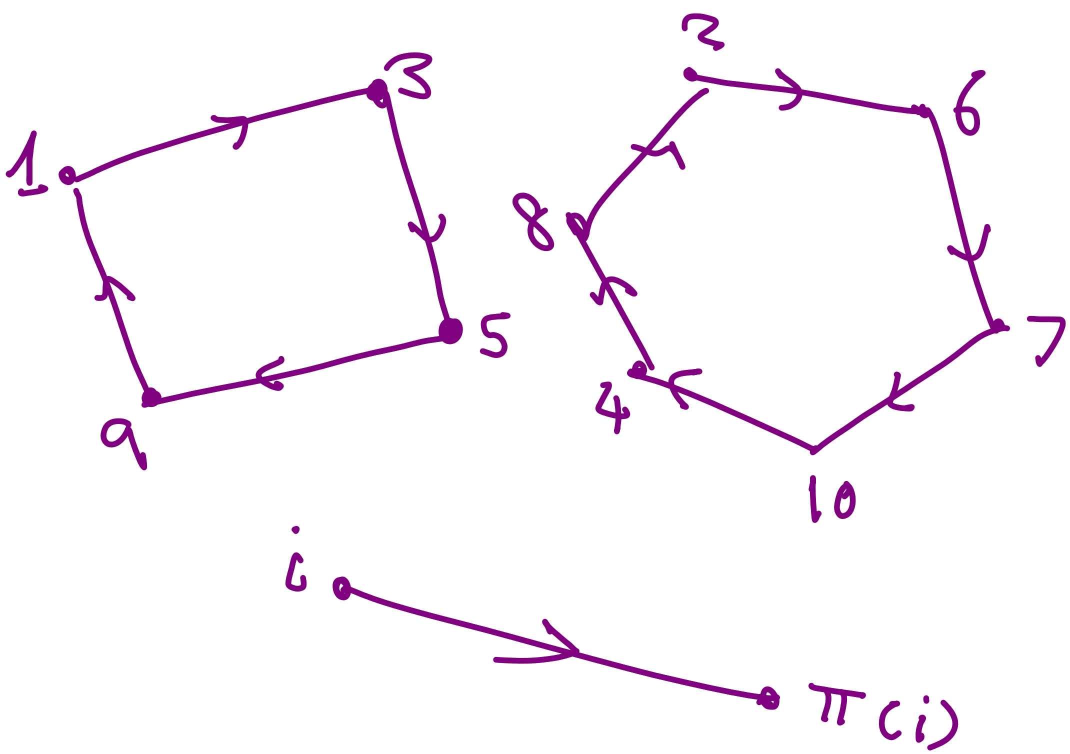
A derangement is permutation π of $\{1, 2, \cdots, n$ such that

$$\pi(i) \neq i \quad , \quad i = 1, 2, \cdots, n$$

$n = 10$

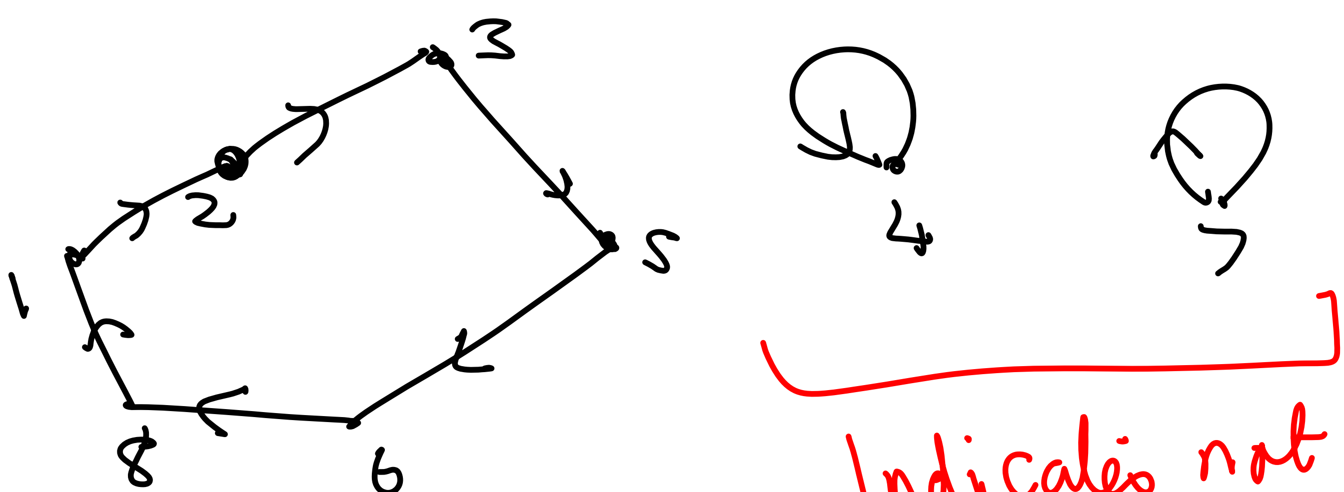
i	1	2	3	4	5	6	7	8	9	10
π	3	6	5	8	9	7	10	2	1	4

A derangement.



Not a derangement

i	1	2	3	4	5	6	7	8
$\pi(i)$	2	3	5	4	6	8	7	1



Indicates not a derangement

To use PIE we have to find the A_i

Each A_i is a set of permulations

derangement is
(not in A_1) and (not in A_2)
and . . . .

$$A_i := \{ \pi : \pi(i) = i \}$$

\# derangements =

$$\sum_{S \subseteq [n]} (-1)^{|S|} |A_S|$$

$$|A_\emptyset| = n!$$

$$A_1 = \begin{array}{c|ccccc} i & 1 & 2 & 3 & \cdots & n \\ \pi(i) & 1 & * & * & \cdots & * \end{array}$$

any permutation
$2, 3, \ldots, n$ here

$$|A_1| = (n-1)!$$

$$A_{\{3,5,6\}} \quad \begin{array}{c|cccccccc} i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ \pi(i) & * & * & 3 & * & 5 & 6 & * & \cdots \end{array}$$

any permutation
$\{1, 2, 4, 7, \ldots\}$

$$|A_{\{3,5,6\}}| = (n-3)!$$

In general

$$|A_S| = (n-|S|)!$$

#derangements =

$$\sum_{S \subseteq [n]} (-1)^{|S|} (n-|S|)! = \sum_{k=0}^{n} \sum_{|S|=k} (-1)^k (n-k)!$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$\to \frac{n!}{e} \quad \text{as } n \to \infty$$