9\25\09

Discrete Probability

Ω - finite or countable

$$P : \Omega \to \mathbb{R}^+$$

$$\sum_{\omega \in \Omega} P(\omega) = 1$$

Probability of ω.

Fair Coin

$\Omega = \{H, T\}$

$Pr(H) = \frac{1}{2} = Pr(T)$

UNIFORM DISTRIBUTION

Dice

$\Omega = \{1, 2, \dots, 6\}$

$P(i) = 1/6 \;, \; i = 1, \dots 6$

Geometric Distribution

$$\Omega = \{1, 2, \ldots \ldots \}$$

$$p(k) = (1-p)^{k-1} p$$

$$\sum_{k} p(k) = 1$$

Rolling two dice

(i) $\Omega = [6]^2$

$P((x_1, x_2)) = \frac{1}{36}, \forall x_1, x_2$

(ii) $\Omega = \{2, 3, 4, \ldots, 12\}$

$P(2) = \frac{1}{36}, \; P(3) = \frac{1}{18}, \ldots$

$A \subseteq \Omega$ is called an event.

$$P(A) = \sum_{\omega \in A} P(\omega)$$

Roll 2 dice. $\Omega = [6]^2$

$A = \{ (x_1, x_2) : x_1 + x_2 = 7 \}$

$$P(A) = \frac{|A|}{36} = 1/6$$

Pennsylvania Lottery

Choose 7 number $I \subseteq [80]$

State randomly choose $J \subseteq [80]$

$|J| = 11$. $\Omega = \binom{[80]}{11}$ - uniform distribution

$WIN = \{I \subseteq J\}$

$$\Pr(WIN) = \frac{|WIN|}{\binom{80}{11}} = \frac{\binom{73}{4}}{\binom{80}{11}}$$

Balls in boxes

m distinguishable balls

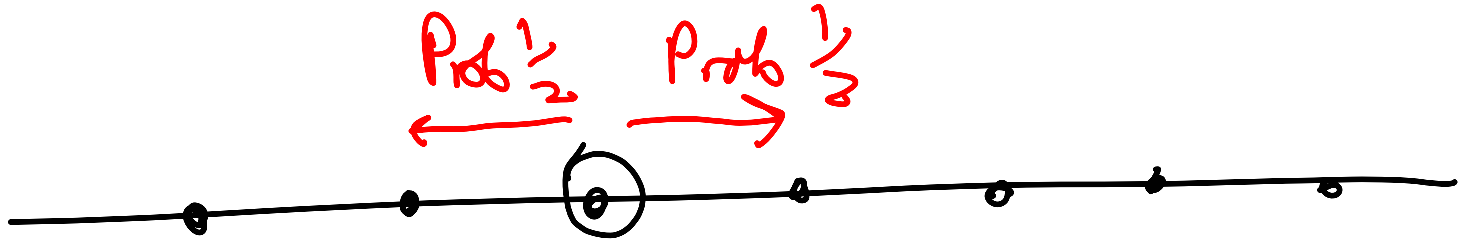
into n distinguishable box

$\Omega = [n]^m$ uniform distribution

$E = \{\text{Box } 1 \text{ is empty}\}$

$$P(E) = \frac{(n-1)^m}{n^m}$$

Random Walk



Make n moves, from 0.

$\Omega = \{L,R\}^n$

$n=6$

LLRLRL

$X_n(\omega)$ = position after n steps.

$n = 2m$

$$\Pr(X_n = 0) = \frac{\binom{n}{m}}{2^n} \approx \sqrt{\frac{2}{\pi n}}$$

Boole's Inequality

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\leq P(A) + P(B)$$

$$P(A_1 \cup A_2 \cup \dots \cup A_n) \leq P(A_1) + P(A_2) + \dots + P(A_n)$$

Coloring Problem

$A_1, A_2, \ldots, A_n \subseteq A$

$|A_i| = k$

Question: can we color the elements of A, Red or Blue so that no A_i is mono-colored.

$A_1 = \{1, 2, 3\}$

$A_2 = \{1, 3, 7\}$

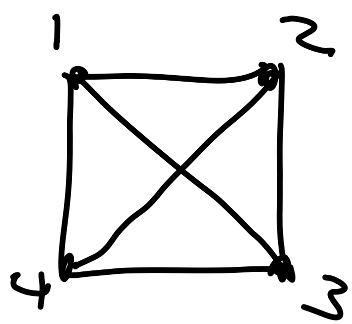
$A_3 = \{2, 5, 6\}$

$A = \{1, 2, 3 - 5, 8\}$

R B B R

$A_1 = \{1, 2\}, \quad A_2 = \{1, 3\}, \quad \ldots \quad A_6 = \{3, 4\}$

$A = \{1, 2, 3, 4\}$

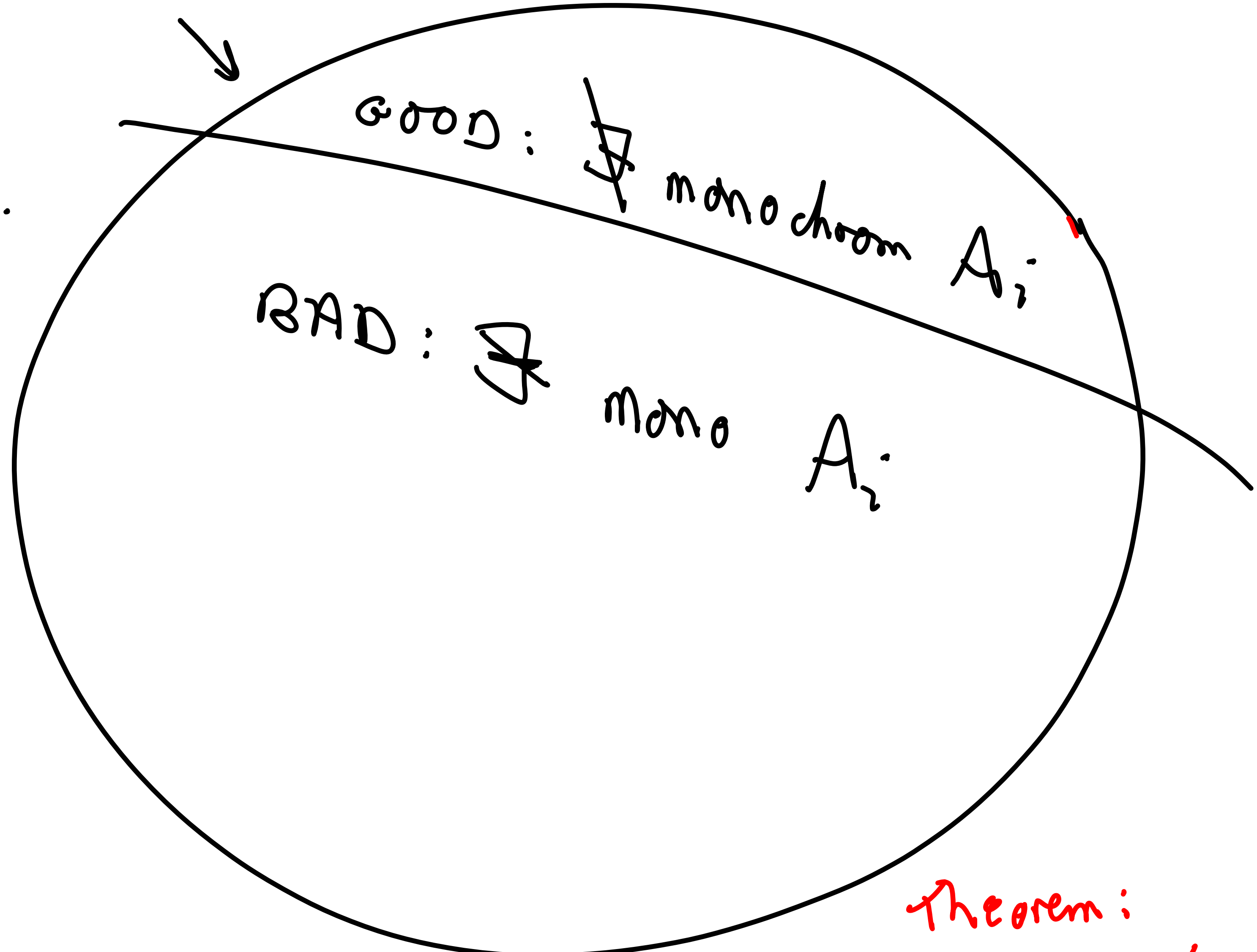


Theorem: Erdős

$n < 2^{k-1} \Rightarrow \exists$ a coloring

Probabilistic Method.

$\Omega = \{2\text{-colorings of } A\}$ $\quad |\Omega| = 2^{|A|}$



Uniform Distribution :

Theorem :

GOOD $\neq \emptyset$

Theorem: $\text{GOOD} \neq \emptyset$

$$\text{BAD} \neq \Omega$$

$$P[\text{BAD}] < 1$$

$\text{BAD}[i] = \{ A_i \leq R \text{ or } A_i \leq B \}$

$$\text{BAD} = \bigcup_{i=1}^{n} \text{BAD}[i]$$

$$P[\text{BAD}] = P\left(\bigcup_{i=1}^{n} \text{BAD}[i]\right)$$

$$\leq \sum_{i=1}^{n} P(\text{BAD}[i])$$

$$= \sum_{i=1}^{n} \frac{1}{2^{k-1}}$$

$$= \frac{n}{2^{k-1}} < 1 \text{ by assumption.}$$