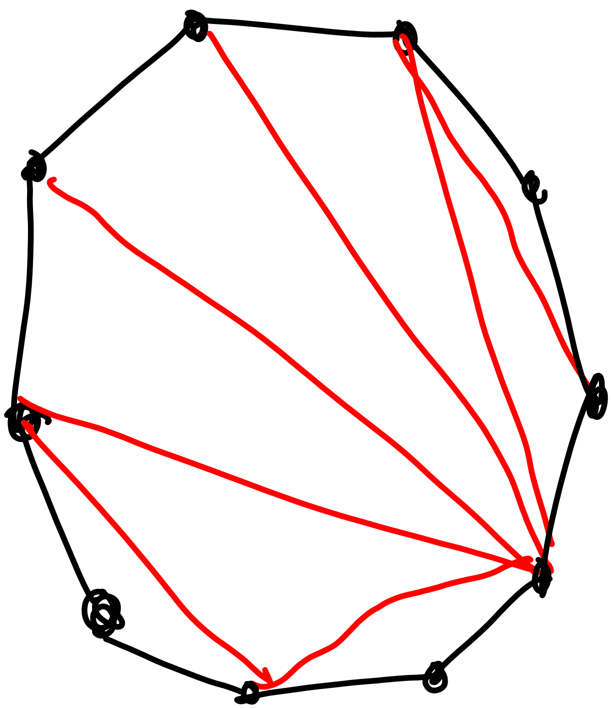
9/23/09

Number of triangulations of an n-gon.



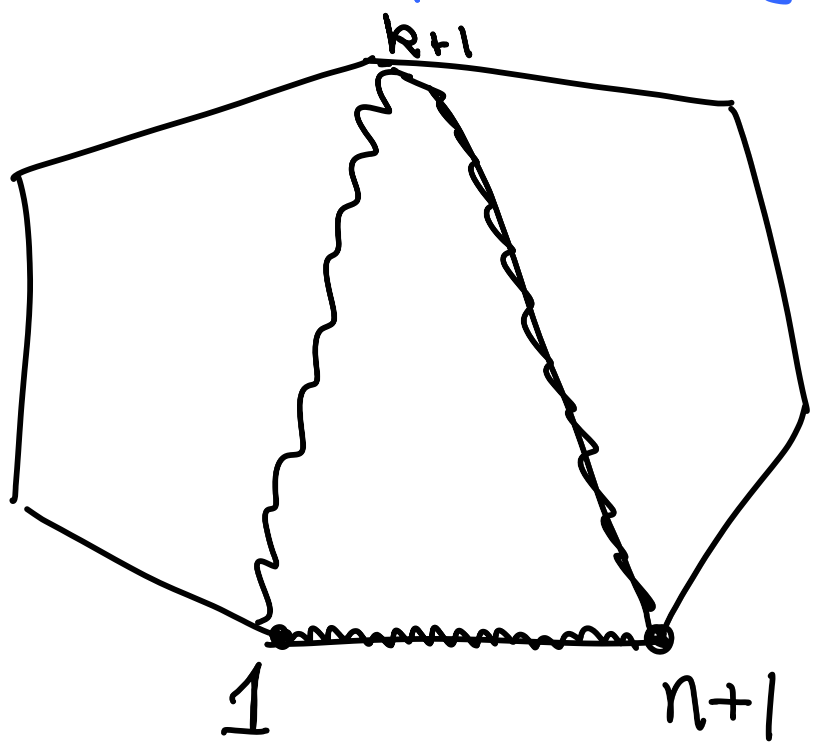
a_n = # of triangulations of P_{n+1}

$\underline{n \geq 2}$

$$a_n = \sum_{k=0}^{n} a_k a_{n-k}$$

similar recurrences appear in many places.

where $a_0 = 0$, $a_1 = 1 = a_2$



$$\underbrace{x + \sum_{n=2}^{\infty} a_n x^n}_{a(x)} = x + \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n$$

$n=0,1$ gives 0 here

[Multiplied by x^n, $n \geq 2$ & Summed]

$a(x)$ is the variable.

x is a coefficient of a quadratic

$$a(x) = x + a(x)^2$$

$$a(x) = \frac{1+\sqrt{1-4x}}{2} \quad \text{or} \quad \frac{1-\sqrt{1-4x}}{2}$$

$x=0$ gives 1 $\qquad$ $x=0$ gives 0

$$a(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x}$$

$$= \frac{1}{2} - \frac{1}{2}(1-4x)^{1/2}$$

binomial theorem

$$= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}\binom{1/2}{n}(-4x)^n$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty}\binom{1/2}{n}(-1)^n 2^{2n} x^n$$

$$= -\sum_{n=1}^{\infty}\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\cdots\left(\frac{1}{2}-n+1\right)}{n!}(-1)^n 2^{2n-1} x^n$$

$$= \sum_{n=1}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}(-1)^n 2^{2n-1} x^n$$

$$= \sum_{n=1}^{\infty} \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\cdots\left(n-\frac{3}{2}\right)}{n!}\cdot 2^{2n-1} x^n$$

$$= \sum_{n=1}^{\infty} \frac{1\times 3\times 5\times\cdots\times 2n-3}{n!} 2^{n-1} x^n$$

$2 \times 4 \quad 6 \quad \times \quad \times 2n-2$

$$= \sum_{n=1}^{\infty} \frac{1\times 3\times 5\times\cdots\times 2n-3}{n!} 2^{n-1} x^n$$

$2\times 4\times 6\times\cdots\times(2n-2)$

$$=\sum_{n=1}^{\infty}\frac{\overset{2}{1}\times\overset{4}{3}\times\overset{6}{5}\times\cdots\times\overset{\times 2n-2}{2n-3}}{n!\quad 2\times4\times6\times\cdots\times(2n-2)}\,2^{n-1}x^{n}$$

$$=\sum_{n=1}^{\infty}\frac{(2n-2)!}{n!\,(n-1)!}\,x^{n}$$

$$=\sum_{n=1}^{\infty}\underbrace{\frac{1}{n}\binom{2n-2}{n-1}}_{a_n}\,x^{n}$$