9\21\09

$$a_n - 5a_{n-1} + 6a_{n-2} = 2^n \qquad n \geq 0$$

$$a_0 = 1, \quad a_1 = 5$$

$$\sum_{n=2}^{\infty} (a_n - 5a_{n-1} + 6a_{n-2}) x^n = \sum_{n=2}^{\infty} 2^n x^n$$

$$\left(a(x) - 1 - 5x\right) - 5x\left(a(x) - 1\right) + 6x^2 a(x) = \frac{1}{1-2x} - 1 - 2x$$

$$\sum_{n=2}^{\infty} a_n x^n = a(x) - a_0 - a_1 x$$

$$a(x) - 1 - 5x - 5x\left(a(x) - 1\right) + 6x^2 a(x) = \frac{1}{1-2x} - 1 - 2x$$

$$a(x)\left(1 - 5x + 6x^2\right) = \frac{1}{1-2x} - 1 - 2x + 1 + 5x - 5x$$

$$= \frac{1}{1-2x} - 2x$$

$$a(x) = \frac{1}{(1-2x)^2(1-3x)} - \frac{2x}{(1-2x)\ (1-3x)}$$

$$= \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1-3x}$$

$$= A\sum_{n=0}^{\infty}(2x)^n + B\sum_{n=0}^{\infty}(n+1)(2x)^n + C\sum_{n=0}^{\infty}(3x)^n$$

$$a_n = A\,2^n + B(n+1)\,2^n + C\,3^n$$

$$a(x) = \frac{1}{(1-2x)^2(1-3x)} - \frac{2x}{(1-2x)(1-3x)}$$

$$= \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1-3x}$$

$$1 - (1-2x) = A(1-2x)(1-3x) + B(1-3x) + C(1-2x)^2$$

$x = \frac{1}{3}$: $\frac{2}{3} = \frac{1}{9}C \Rightarrow C = 6$

$x = \frac{1}{2}$: $1 = -\frac{1}{2}B \Rightarrow B = -2$

$x = 0$: $0 = A + B + C \Rightarrow A = -4.$

$$a(x) = \sum_{n=0}^{\infty} a_n x^n \quad \&$$

$$b(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$a(x)\,b(x) = (a_0 + a_1 x + a_2 x^2 + \cdots\cdots)$$

$$\times\ (b_0 + b_1 x + b_2 x^2 + \cdots\cdots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)\,x + (a_0 b_2 + a_1 b_1 + a_2 b_0)\,x^2 +$$

$$\cdots \quad + \left(\underbrace{\sum_{k=0}^{n} a_k b_{n-k}}_{c_n}\right) x^n + \cdots$$

Derangements

Let d_n = # of derangements

$$\underset{\#\text{ permutations}}{n!} = \sum_{k=0}^{n} \underbrace{\binom{n}{k} d_{n-k}}_{\substack{\#\text{ permutations with} \\ \text{exactly } k \text{ "fixed points"}}}$$

i is fixed if $\pi(i) = i$

$\binom{n}{k}$ choices for the fixed points

d_{n-k} ways of permuting rest, without adding fixed point

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k} \quad ; n \geq 0$$

Divide through by $n!$

$$1 = \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{d_{n-k}}{(n-k)!}$$

$$a_k = \frac{1}{k!} \qquad b_{n-k} = \frac{d_{n-k}}{(n-k)!}$$

$$a(x) = e^{x} \qquad b(x) = \sum_{n=0}^{\infty} \frac{d_n}{n!} x^n$$

$$\sum \left(1 = \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{d_{n-k}}{(n-k)!} \right) \times x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{d_{n-k}}{(n-k)!} \right) x^n$$

$$= e^x d(x)$$

$$d(x) = \sum_{n=0}^{\infty} \frac{d_n}{n!} x^n$$

[Exponential Generating Function]

$$\frac{1}{1-x} = e^{x}\, d(x)$$

$$d(x) = e^{-x} \cdot \frac{1}{1-x}$$

$$\begin{array}{cc} a(x) & b(x) \end{array}$$

$$\frac{d_n}{n!} = \sum_{k=0}^{n} a_k \, b_{n-k}$$

$$a_k = \frac{(-1)^k}{k!} \qquad b_{n-k} = 1$$

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \, .$$