8/26/09

$\phi_{1-1}(m,n)$ = # of 1-1 mappings of $[n]$ to $[m]$

= # sequences

$a_1 a_2 \ldots a_n$ $\quad$ $[a_i \in [m]]$

and $a_i \neq a_j$, $i \neq j$

a_1 — m choices

a_2 — $m-1$ choices

a_3 — $m-2$ choices

$\vdots$

a_n — $m-n+1$ choices

$\#$choices $= m(m-1)\cdots(m-n+1)$

Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

n choose k

X is a finite set.

$$\binom{X}{k} := \{k\text{-subsets of } X\}$$

$$\left|\binom{X}{k}\right| = \binom{|X|}{k}$$

<u>Proof</u>

$n = |X|$

$$\underbrace{k!}_{\text{order them}} \left|\underbrace{\binom{X}{k}}_{\text{choose elements}}\right| = \underbrace{n(n-1)\cdots(n-k+1)}_{\text{sequences of } k \text{ distinct members}}$$

"Pirates and Gold"

m, n are non-negative integers

$$S(m,n) = \{ (i_1, i_2, \ldots, i_n) \in \mathbb{Z}_+^n : i_1 + i_2 + \cdots + i_n = m \}$$

n pirates

m pieces of gold.

Theorem

$$|S(m,n)| = \binom{m+n-1}{n-1}$$

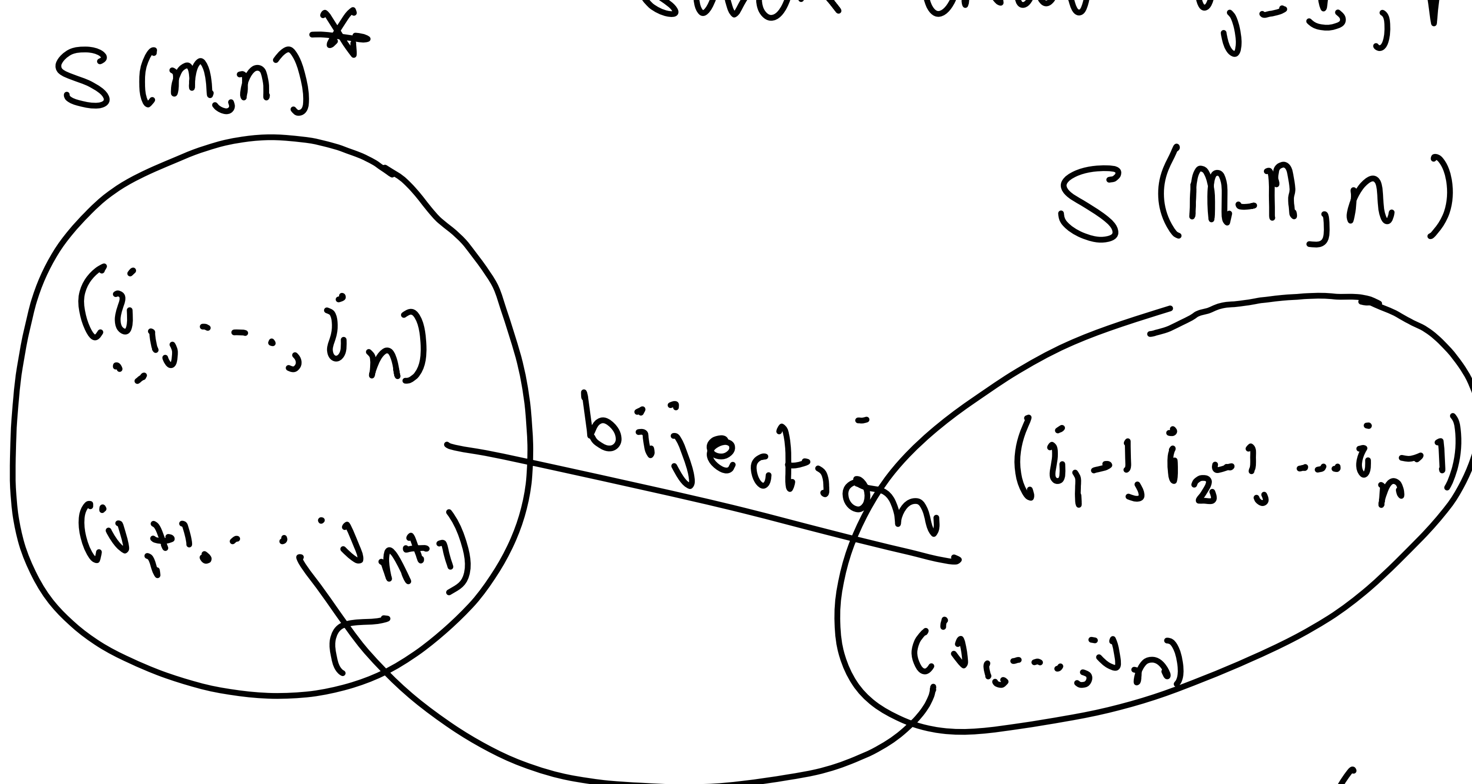
Proof

$$\overbrace{\underbrace{*\ \ *}_{i_1=2}\ \ *\ \ \underbrace{*\ \ *\ \ *}_{i_2=3}\ \ *\ \underbrace{\ }_{i_3=0}\ *\ \ \underbrace{*\ \ *\ \ *}_{i_4=3}}^{m+n-1}$$

$n=4$

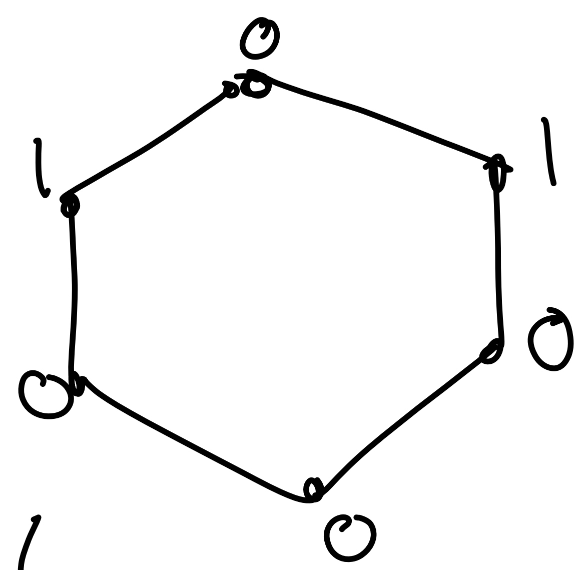
$\square$

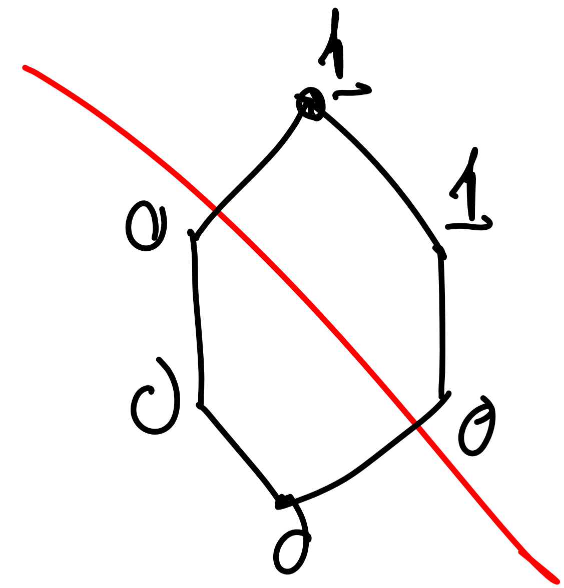
$$S(m,n)^* = \{(i_1, \dots, i_n) \in S[m,n] \text{ such that } i_j \geq 1, \forall j\}$$



$$|S(m-n,n)| = \binom{m-1}{n-1}$$

Placing 1's on a cycle

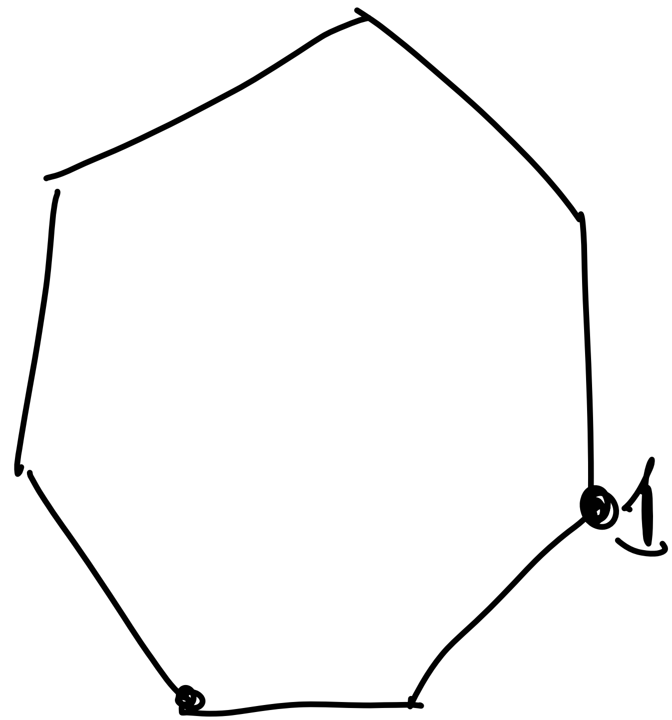




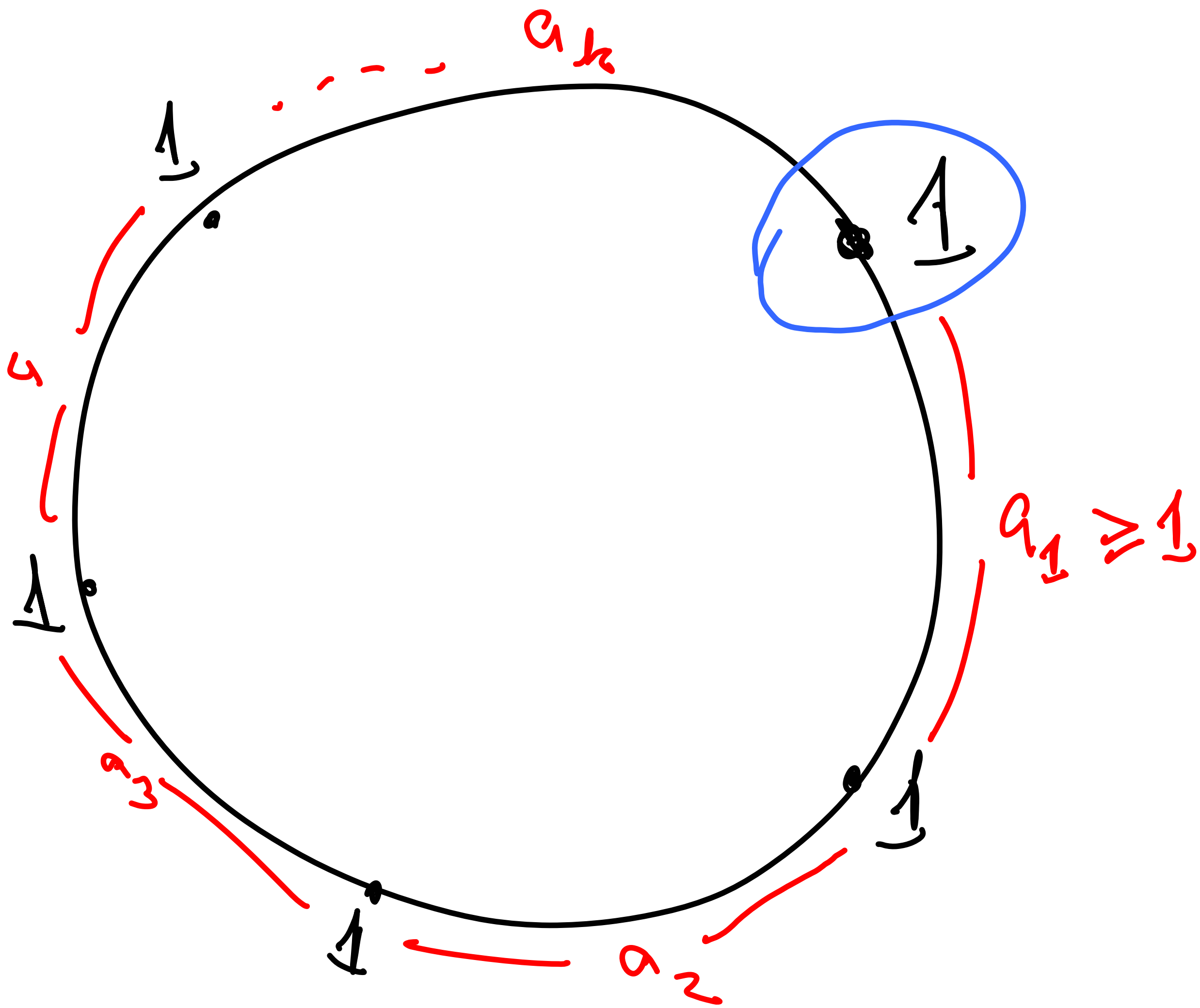
$n = 6$

$k = 2$

less than $\binom{n}{k}$.



Choose a position (in n ways) and put a $\mathbb{1}$ there



$$a_1 + a_2 + \cdots + a_k = n - k$$

$$\# \text{ chores} = n \times \binom{n-k-1}{k-1} / k.$$

Identities

(1) $\binom{n}{r} = \binom{n}{n-r}$

choose set — choose complement

or

do it algebraically

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Include object $n+1$ | Exclude object $n+1$ | Choose $k+1$ from $n+1$

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} \quad (*)$$

<u>Proof</u>

First proof by Induction.

$S(n) \equiv (*)$ is true, for all $k \leq n$.

Induction on n.

Base Case: $n = 1$ — $k = 0$

$k = 1$

Confirm $S(n+1)$:

Choose $k \leq n+1$

$$\sum_{m=k}^{n+1} \binom{m}{k} = \sum_{m=k}^{n} \binom{m}{k} + \binom{n+1}{k}$$

induction

$$= \binom{n+1}{k+1} + \binom{n+1}{k}$$

Pascal Δ

$$\stackrel{?}{=} \binom{n+2}{k+1}$$