11\20\09

X, group G.

$\{\text{colorings}\}$

Orbit of $x \in X = \{g \circ x : g \in G\}$

Use

$$\#\text{orbits} = \sum_{x \in X} \frac{1}{|O_x|}$$

(simplified)

$x \in X$

S_x = stabiliser of x

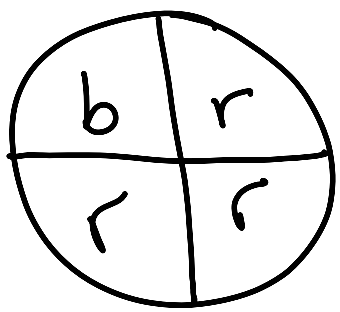
$= \{ g : g \circ x = x \}$

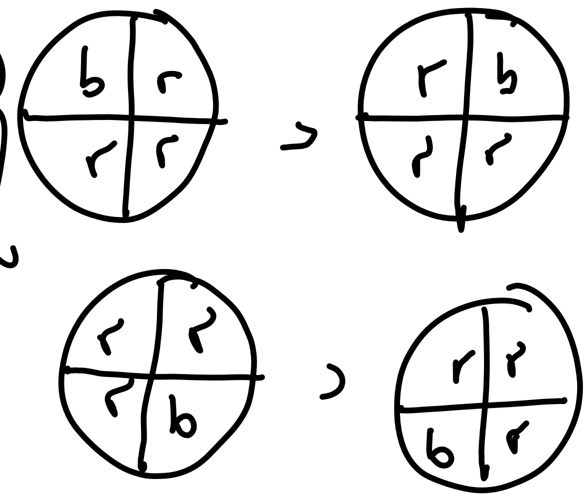
= { actions, that do not change coloring x }

Lemma

$$|O_x||S_x| = |G|$$

Example

$x =$ 

, $O_x = \{$ 

$\}$

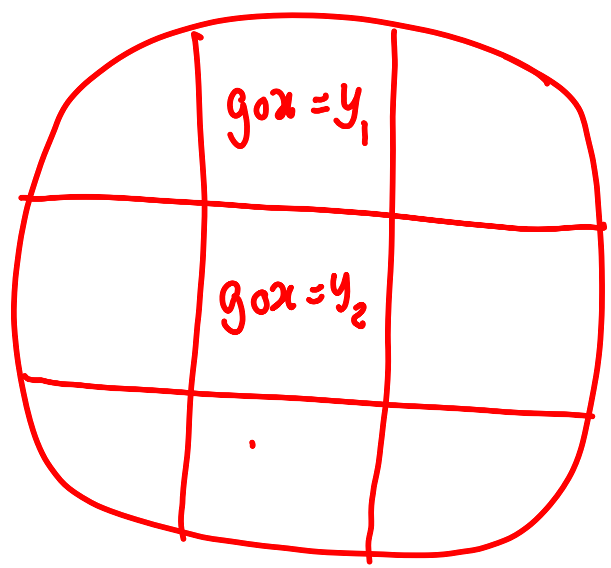
$$S_x = \{e\}$$

Proof

Fix $x \in X$ and define an equivalence relation $\sim$ on G.

$$g_1 \sim g_2 \text{ iff } g_1 \circ x = g_2 \circ x$$

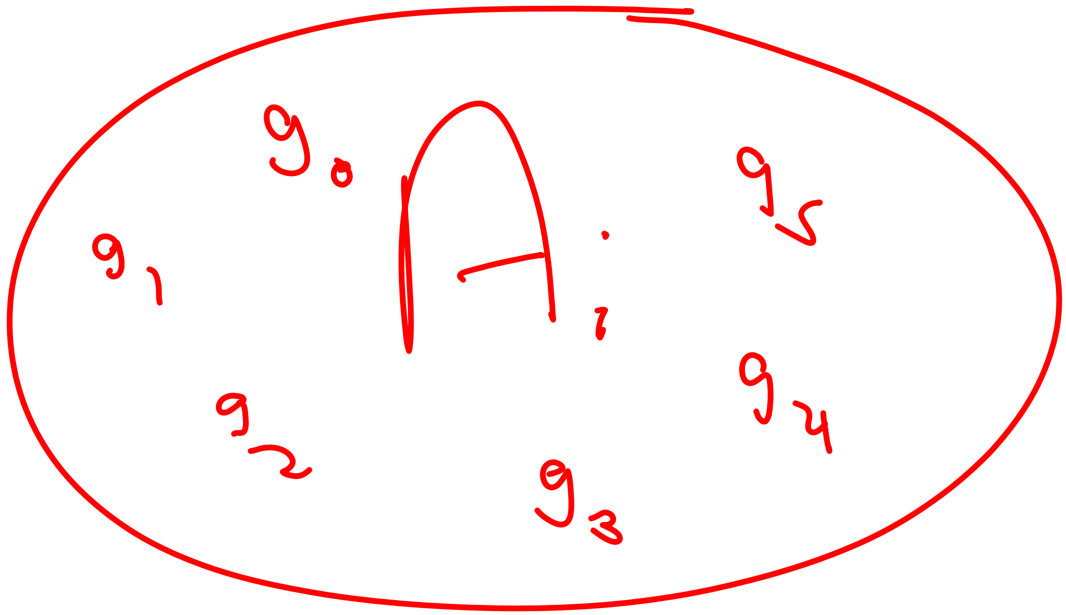
Equivalence classes are
$A_1, A_2, \ldots A_m$



(i) $m = |O_x|$

(ii) $|A_i| = |S_x|$

(ii) $|A_i| = |S_x|$



$g_0 \circ x = g_i \circ x \quad \forall i$

iff $(g_0^{-1} g_i) \circ x = x$ iff $g_0^{-1} g_i \in S_x$

iff $g_i \in g_0 S_x$. & $|g_0 S_x| = |S_x|$.

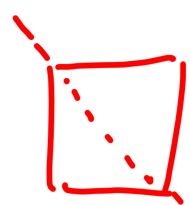
Example 2

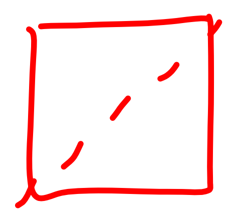
$$x = \begin{array}{|c|c|} \hline r & b \\ \hline b & r \\ \hline \end{array}$$

$$O_x = \left\{ \begin{array}{|c|c|} \hline r & b \\ \hline b & r \\ \hline \end{array} \, , \, \begin{array}{|c|c|} \hline b & r \\ \hline r & b \\ \hline \end{array} \right\}$$

$$S_x = \{ e, \; b, \; r, \; s, \; \ldots \}$$

180° rotation





$$\#\,\text{orbits} = \sum_{x \in X} \frac{1}{|O_x|} \leftarrow \frac{|G|}{|S_x|}$$

$$= \sum_{x \in X} \frac{|S_x|}{|G|}$$

$$= \frac{1}{|G|} \sum_{x \in X} |S_x|.$$

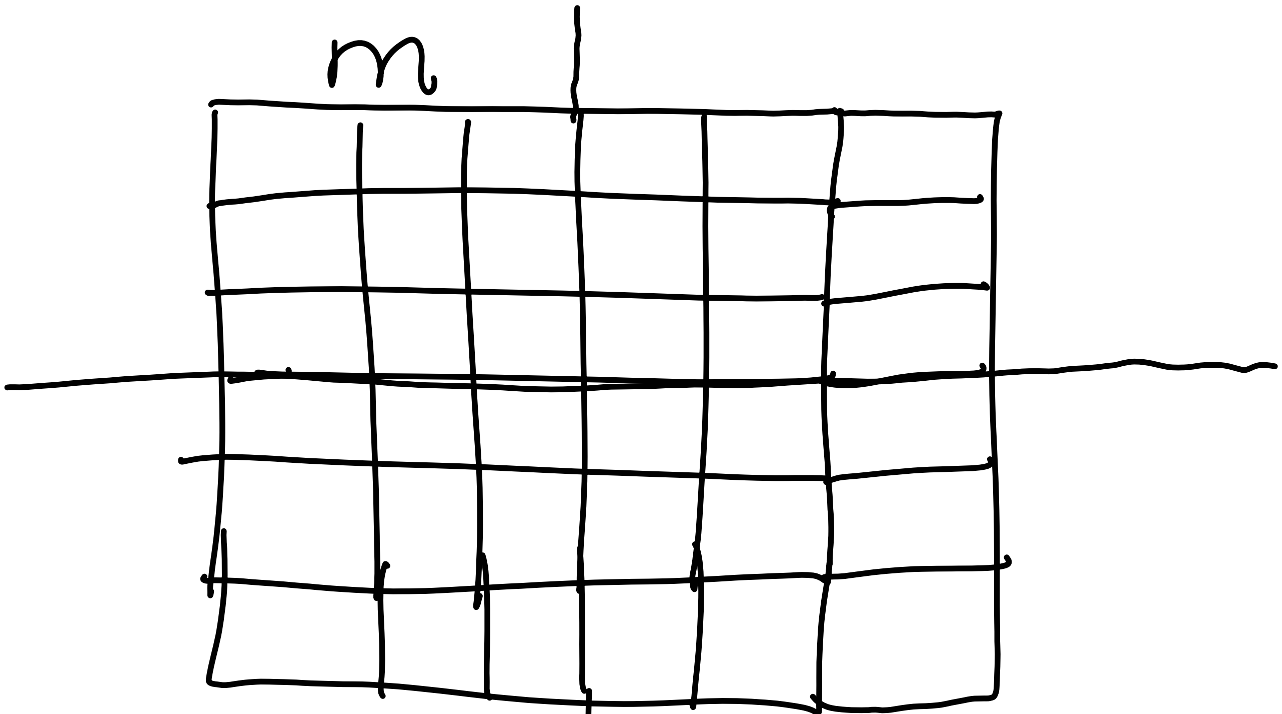
Theorem Burnside's Lemma

$$\#\text{orbits} = \frac{1}{|G|} \sum_{x \in X} |S_x|$$

$$= \frac{1}{|G|} \sum_{x \in X} \sum_{g \in G} \mathbb{1}_{\{g \circ x = x\}}$$

$$= \frac{1}{|G|} \sum_{g \in G} \underbrace{\sum_{x \in X} \mathbb{1}_{\{g \circ x = x\}}}_{\#\text{ of } x \text{ s.t. } g \circ x = x}$$

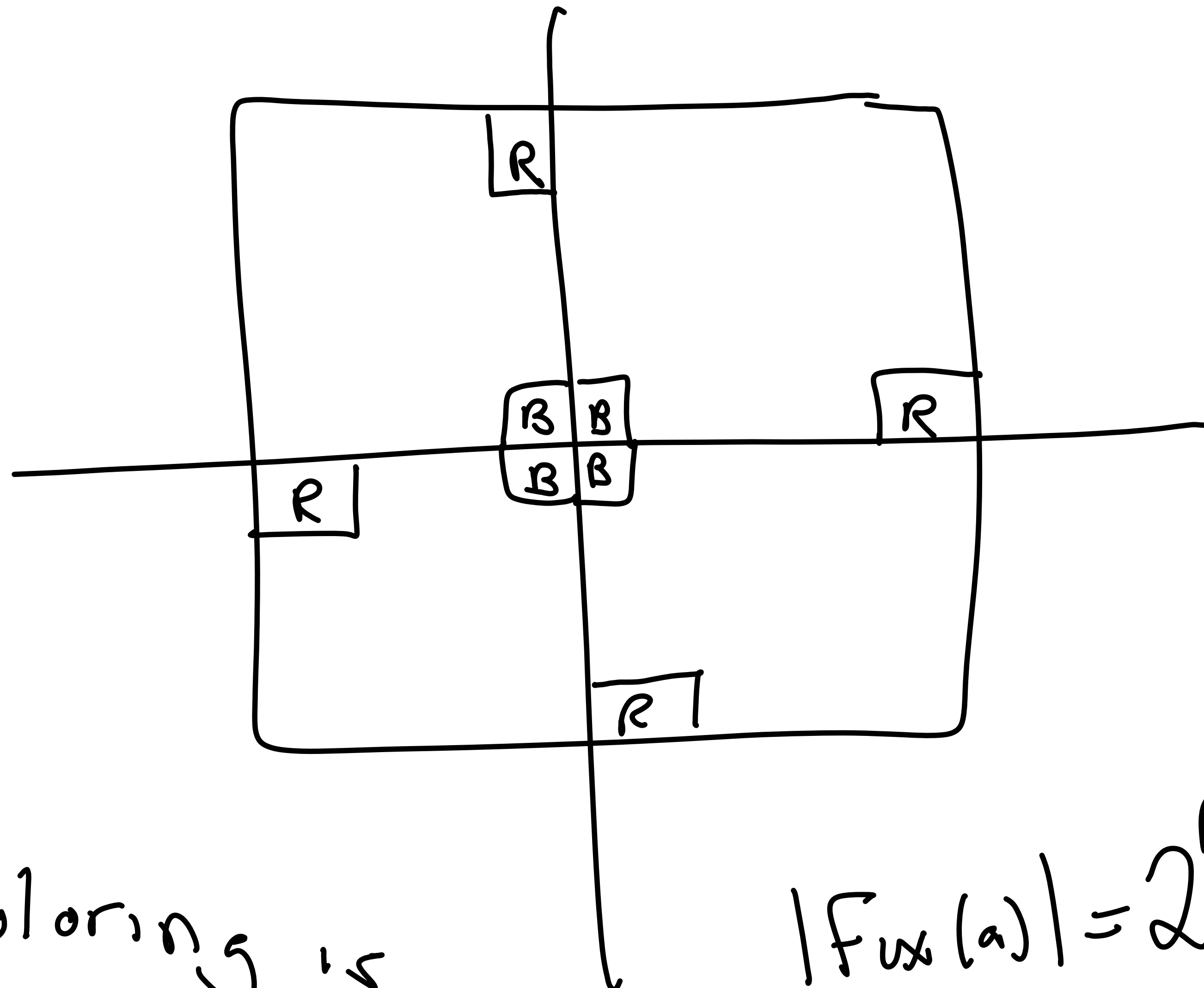
$|\mathrm{Fix}(g)| = \{x : g \circ x = x\}$



$2m \times 2m$ chess board.

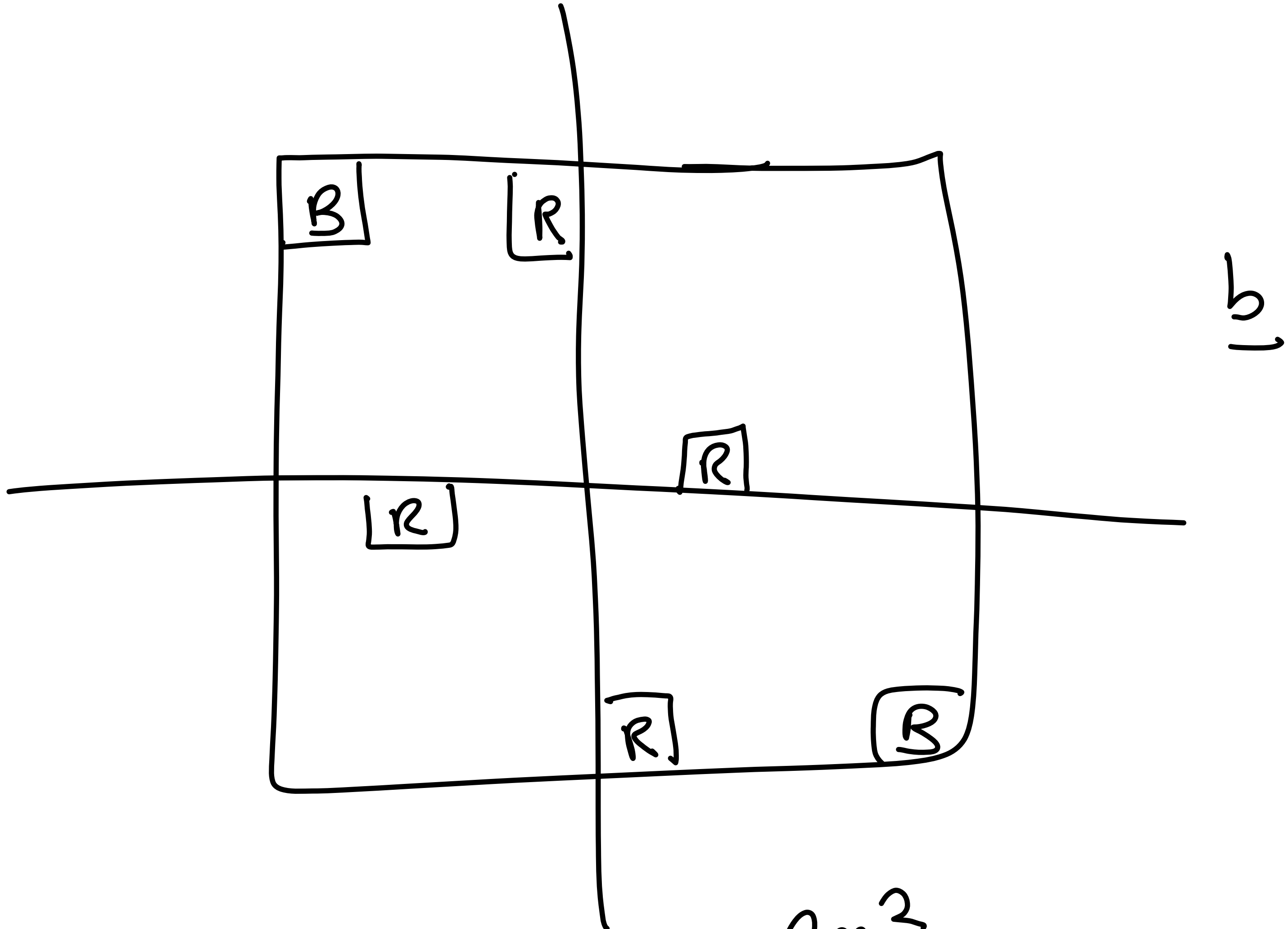
g	e	a	b	c	p	q	r	s
$\lvert \mathrm{Fix}(g) \rvert$	2^{4m^2}	2^{m^2}	2^{2m^2}	2^{m^2}	2^{2m^2}	2^{2m^2}	$2^{m(2m+1)}$	$2^{m(2m+1)}$

$$\#\text{orbits} = \frac{1}{8}\left(\qquad \right)$$

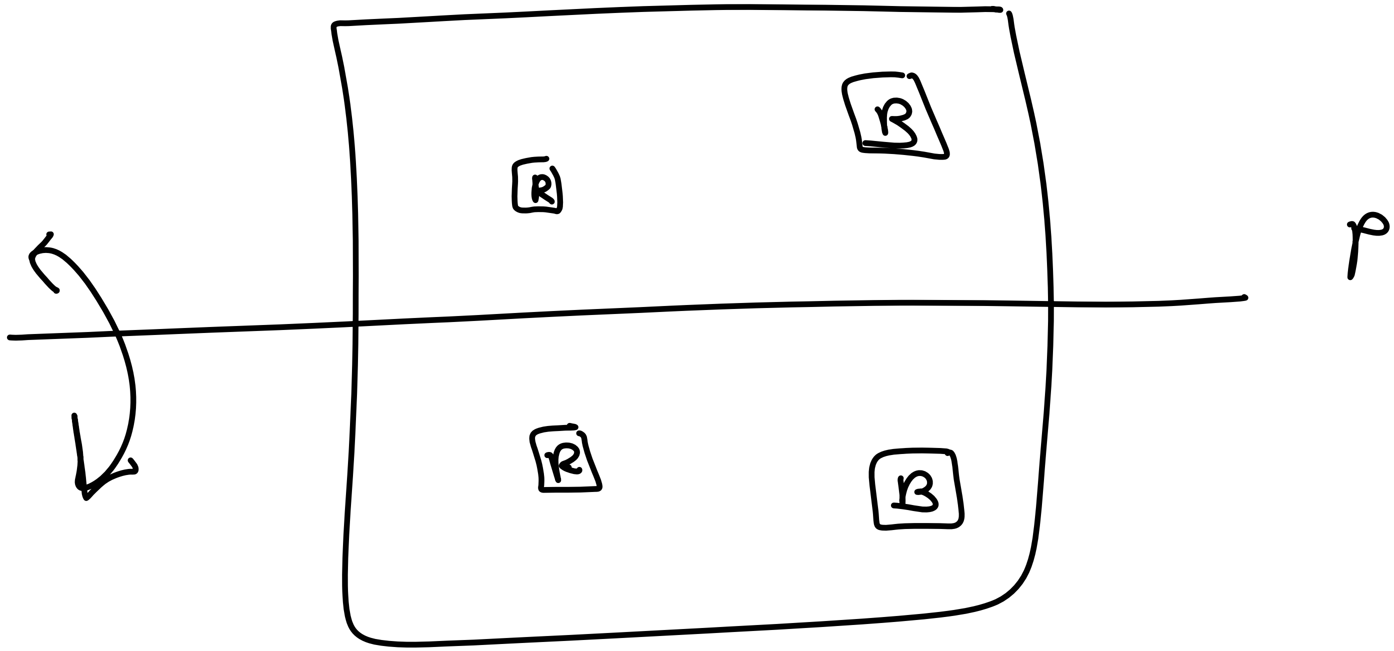


Coloring is in Fix(a)

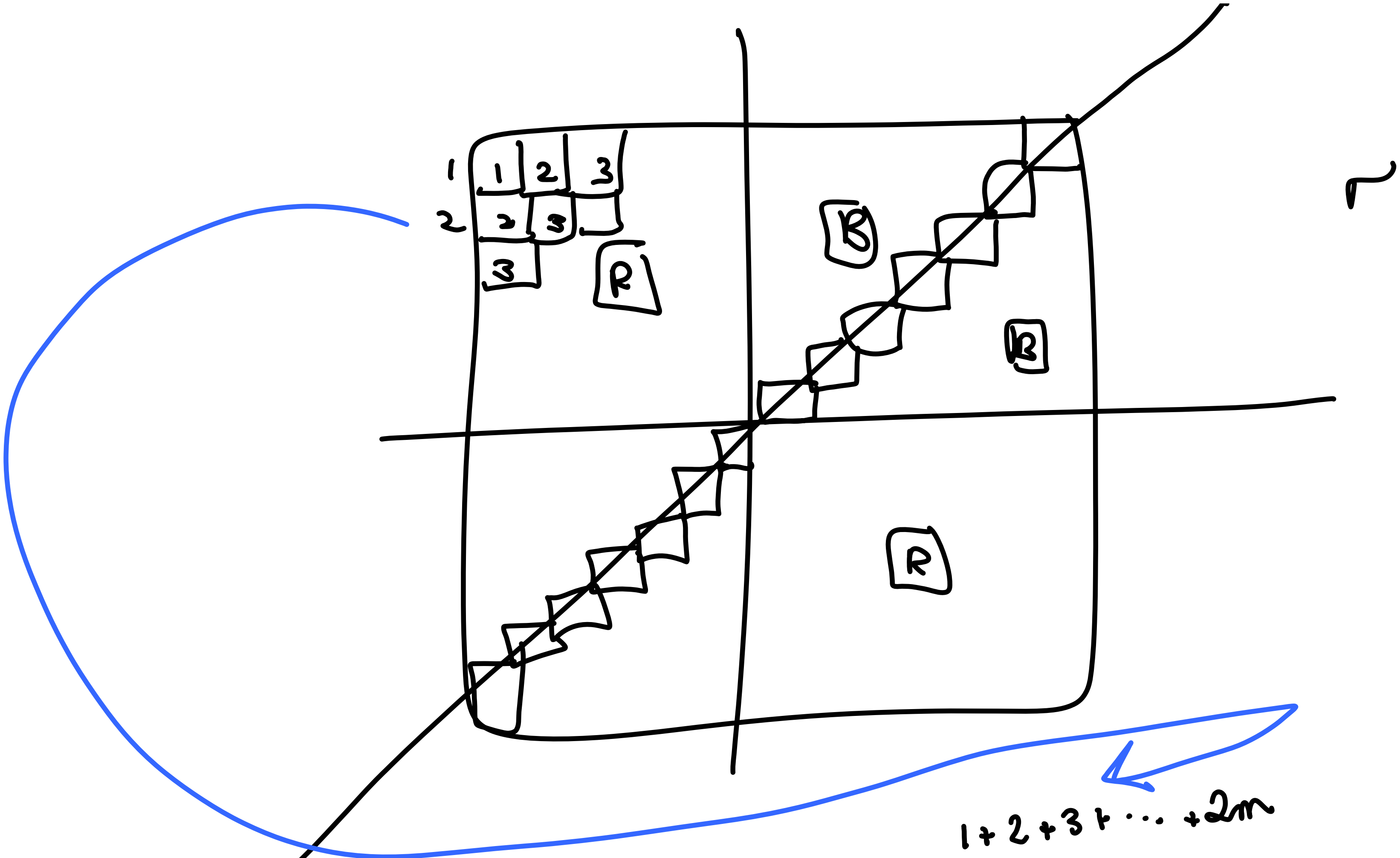
$|\mathrm{Fix}(a)| = 2^{m^2}$



$$|\mathrm{Fix}(b)| = 2^{2n^2}$$



$$|F_{2n}(p)| = 2^{2m^2}$$



$$|F_\alpha(r)| = 2^{1+2+3+\cdots+2m}$$

$$= 2^{2m(2m+1)/2}$$