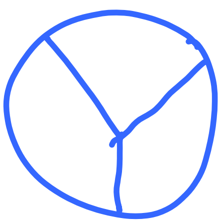
# 11/18/09

## Counting under symmetry.

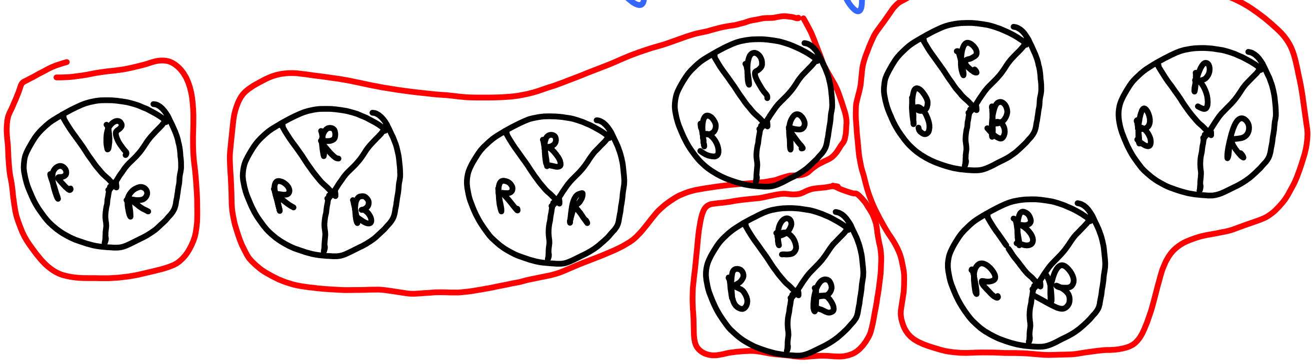


← color each sector Red or Blue

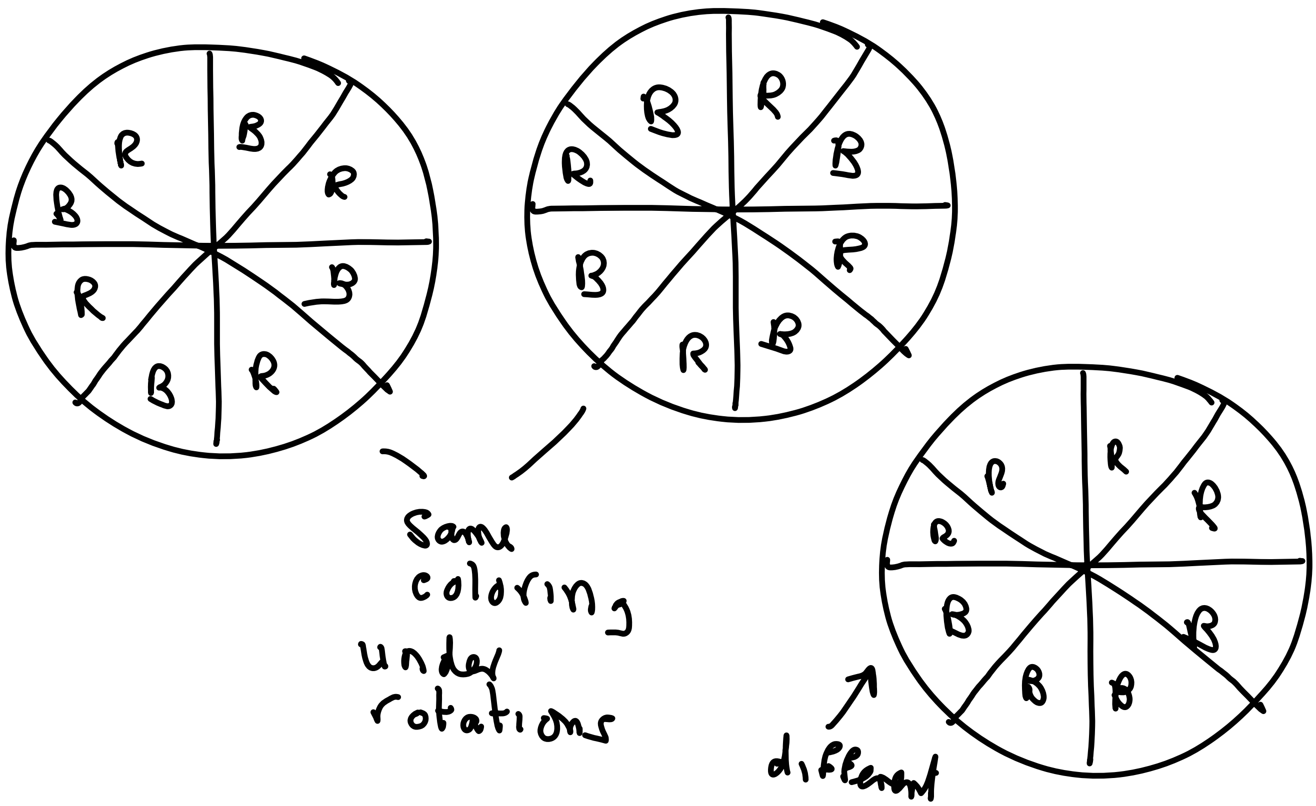
Disc can be rotated.

How many colorings?

4 ways

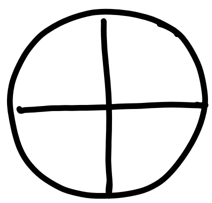


More sectors, more complicated



Formally.

We have a set $X = \{$ colorings of an object $\}$

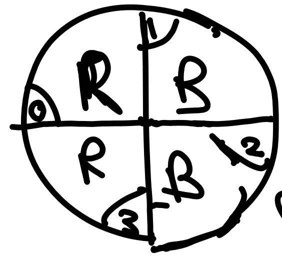


$|X| = 2^4$

There is a group $G$ of permutations of $X$.

Given $g \in G$ and $x \in X$ we get $g(x)$

Example




$|X| = 2^4$

$G = \{$rotations of $\}$

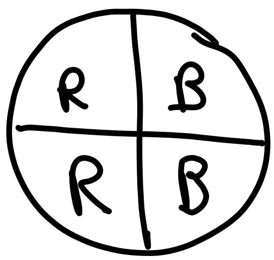
| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $g_0$ | 0 | 1 | 2 | 3 |
| $g_1$ | 1 | 2 | 3 | 0 |
| $g_2$ | 2 | 3 | 0 | 1 |
| $g_3$ | 3 | 0 | 1 | 2 |

← do nothing

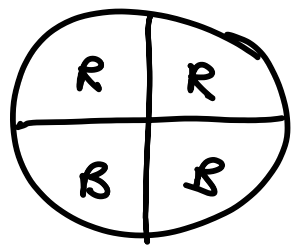
$G = \{g_0, g_1, g_2, g_3\}$

For each $x \in X$ and $g \in G$ we have $g(x) \in X$.

$g$: colorings $\longrightarrow$ colorings

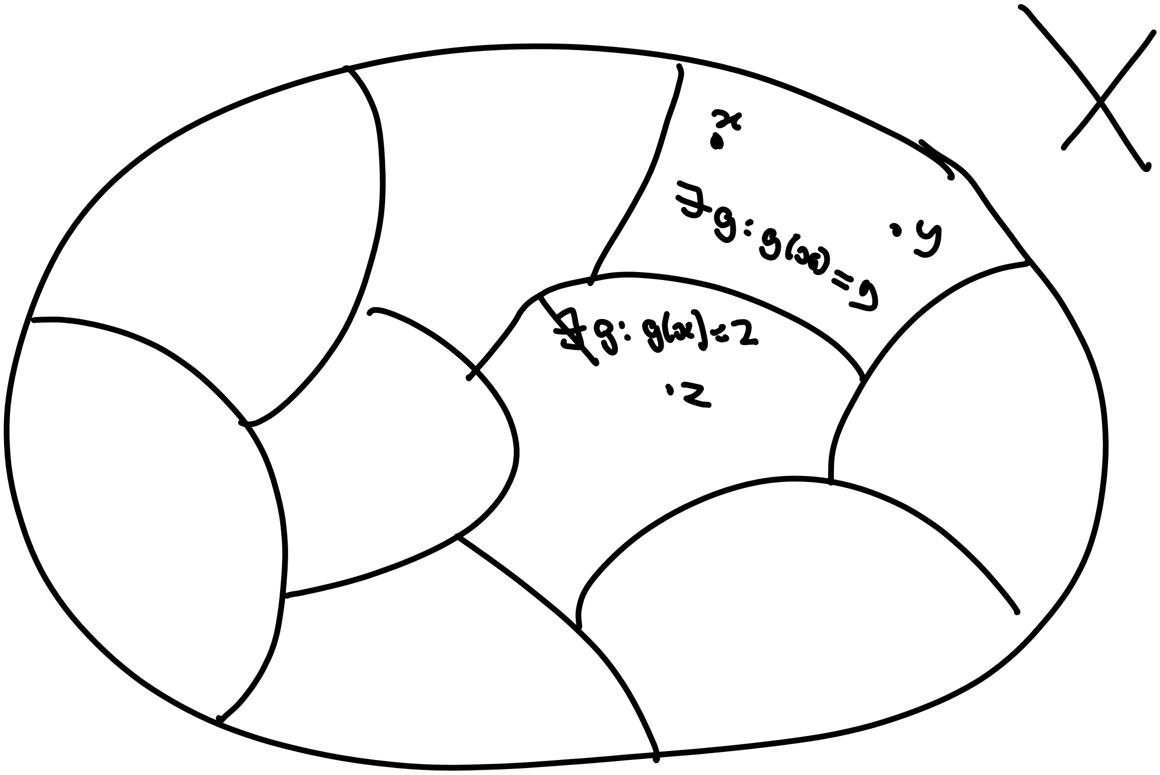



$x$




$g_1(x)$

Two colorings $x, x'$ are same if $\exists g: g(x) = x'$



X
x
∃g : g(x) = y
y
∃g : g(x) = z
z

Structure:

(I) $g_1 \circ g_2(x) = g_1(g_2(x))$

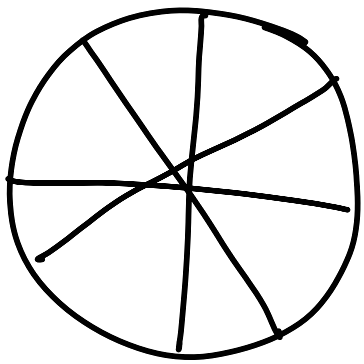
(II) $G$ is a group.

(a) $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$

(b) $\exists \mathbb{1}_X$ s.t. $\mathbb{1}_X(x) = x \quad , \quad \forall x$

(c) $\exists$ inverse $g^{-1}$ s.t. $g^{-1} \circ g = \mathbb{1}_X$

## Example 1



n sectors

Allowed to rotate.

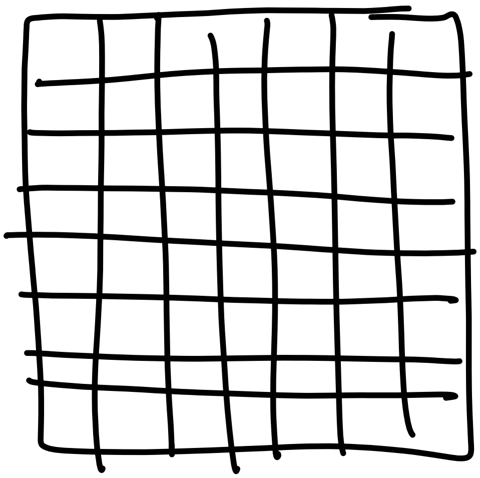
(i) $|X| = 2^n$

(ii) $G = (e_0, e_1, \ldots, e_{n-1})$

$e_j * x \equiv$ coloring you get by rotating through $2\pi j/n$

$n=8$; $x = (R, R, B, R, B, B, R, B)$

$j=3$; $e_j * x = (B, R, B, R, R, B, R, B)$



n×n chessboard

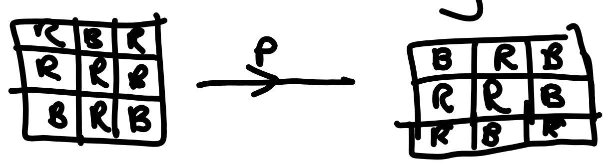
$$|X| = 2^{n^2}$$

$G = \{ e,$ ← identity

$a, b, c$ ← rotations

$p, q,$ ← reflections
hor, vert

$r, s$ ← diagonal



Example $n=2$, diagram on p7

$c$ = rotate $270°$

$$rrbb = \begin{array}{|c|c|} \hline R & R \\ \hline B & B \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline R & B \\ \hline R & B \\ \hline \end{array} = rbbr$$

# Orbits of $G$

If $x \in X$ then the orbit of $x$, $O_x$:

$$O_x = \{ y \in Y : \exists g \in G,\ g * x = y \}$$

Claim: the orbits partition $x$.

(i) $x \in O_x$, $e \in G$.

(ii) Suppose that $z \in O_x \cap O_y$

$$z = g_1 * x = g_2 * y \Rightarrow x = (g_1^{-1} \circ g_2) * y$$

So $g * x = (g \circ (g_1^{-1} \circ g_2)) * y$ and $O_x \subseteq O_y$.

Similarly $O_y \subseteq O_x$.