11\16\09

Sprague-Grundy Numbering:

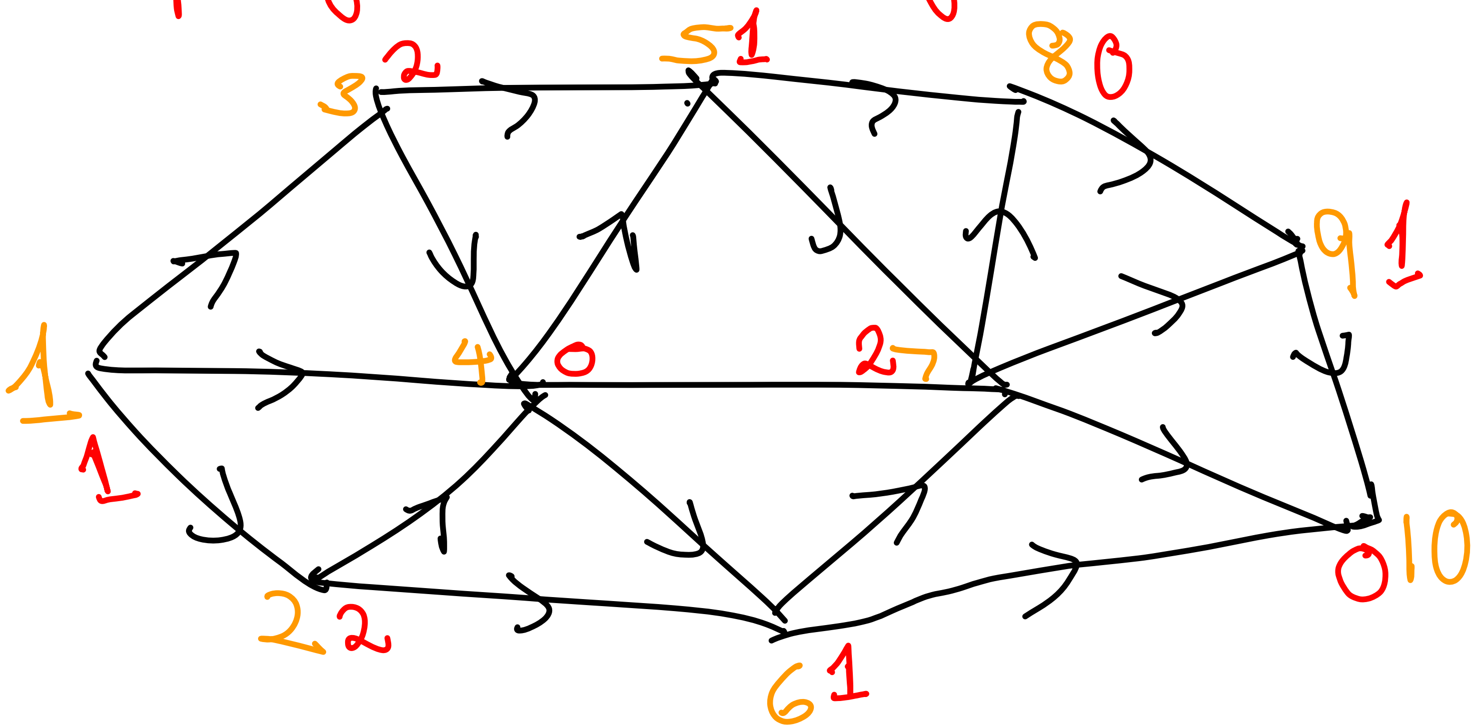
$S \subseteq \{0, 1, 2, \ldots\}$ then

$$\text{mex}(S) = \min\{x \geq 0 : x \notin S\}$$

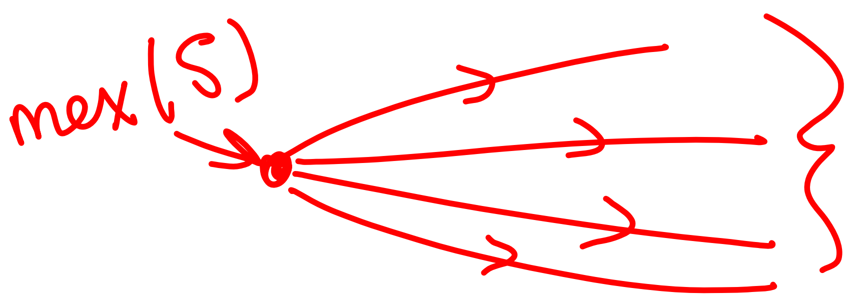
$S = \{0, 1, 3, 4\}$ $\quad$ $\text{mex}(S) = 2$

$S = \{3, 12, 13\}$ $\quad$ $\text{mex}(S) = 0$

Topological numbering



SG-numbering



$S = N^{+}(S)$

Claim

$$x \in P \longleftrightarrow g(x) = 0$$

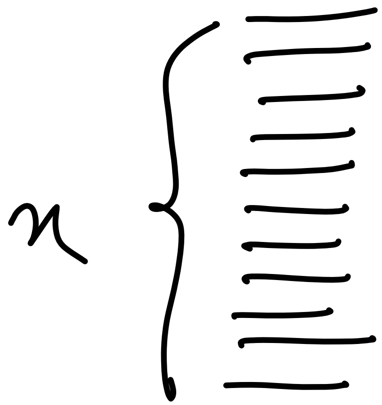
Proof

$$x \in N \quad \text{iff} \quad N^+(x) \cap P \neq \emptyset$$

To show $\|\|$

$$g(x) > 0 \quad \text{iff} \quad \exists y \in N^+(y) \text{ s.t. } g(y) = 0$$

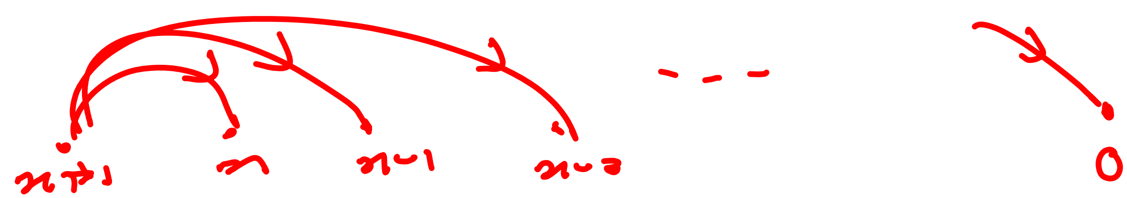
Simplest one pile game

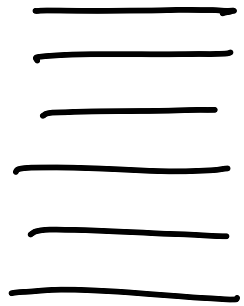


Can take any or all:

$$g(x) = x$$

Induction: true for $x = 0$





One pile

(i) Remove 2, 4, 6, 8, . . . but NOT whole pile

(ii) Whole pile if there are an odd number in the pile.

$g(x) = ?$

x	0	1	2	3	4	5	6	7	8	9	10
$g(x)$	0	1	0	2	1	3	2	4	3		

$$g(2k) = k-1$$
$$g(2k-1) = k$$
$\Big\}$ assume true for some k

$$g(2k) = \text{mex}\{\underset{k-2}{g(2k-2)}, \underset{k-3}{g(2k-4)}, \ldots, \underset{0}{g(2)}\}$$

$$= k-1$$

$$g(2k-1) = \text{mex}\{\underset{k-1}{g(2k-3)}, \underset{k-2}{g(2k-5)}, \ldots, g(1), g(0)\}$$

$$= k$$

SG numbering for the sum of games.

2 games

	Game 1	Game 2	$G_1 + G_2$
Position	$(x_1,$	$x_2)$	
	$g_1(x_1)$	$g_2(x_2)$	$g_1(x_1) \oplus g_2(x_2)$

Integer a, b, $a \oplus b$ is defined as follows:

$$a = a_m \, a_{m-1} \cdots a_1 \, a_0 \quad \leftarrow \text{binary}$$

$$b = b_m \, b_{m-1} \cdots b_1 \, b_0$$

$$a \oplus b \qquad \cdots \; a_1 \oplus b_1 \; a_0 \oplus b_0$$

$1 \oplus 1 = 0$

$0 \oplus 0 = 0$

$1 \oplus 0 = 1$

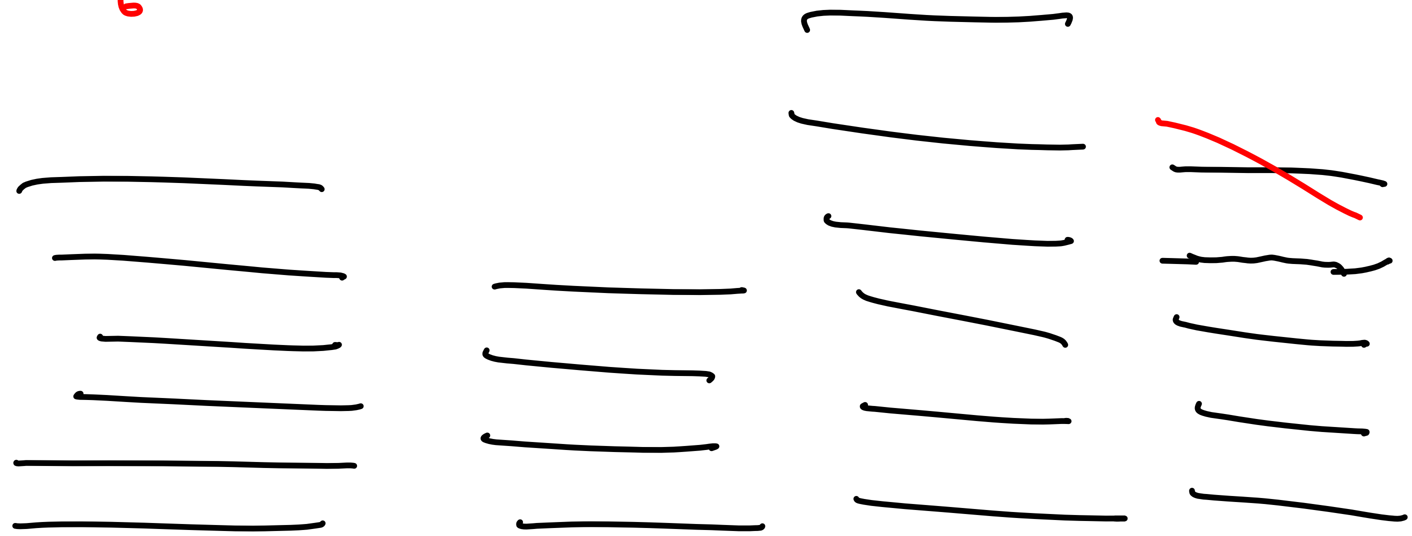
$0 \oplus 1 = 0$

$$37 \oplus 45$$

$$37 = 1\ 0\ 0\ 1\ 0\ 1$$
$$45 = 1\ 0\ 1\ 1\ 0\ 1$$

$$37 \oplus 45 = 0\ 0\ 1\ 0\ 0\ 0$$
$$= 8$$

6



?? Is this a winning position?

$$\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ \hline 0 & 0 & 1 \end{array}$$

winning position