11\13\09

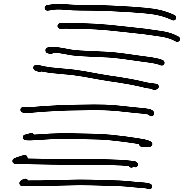
Represent game by a digraph

$D = (X, A)$. ← "Board"

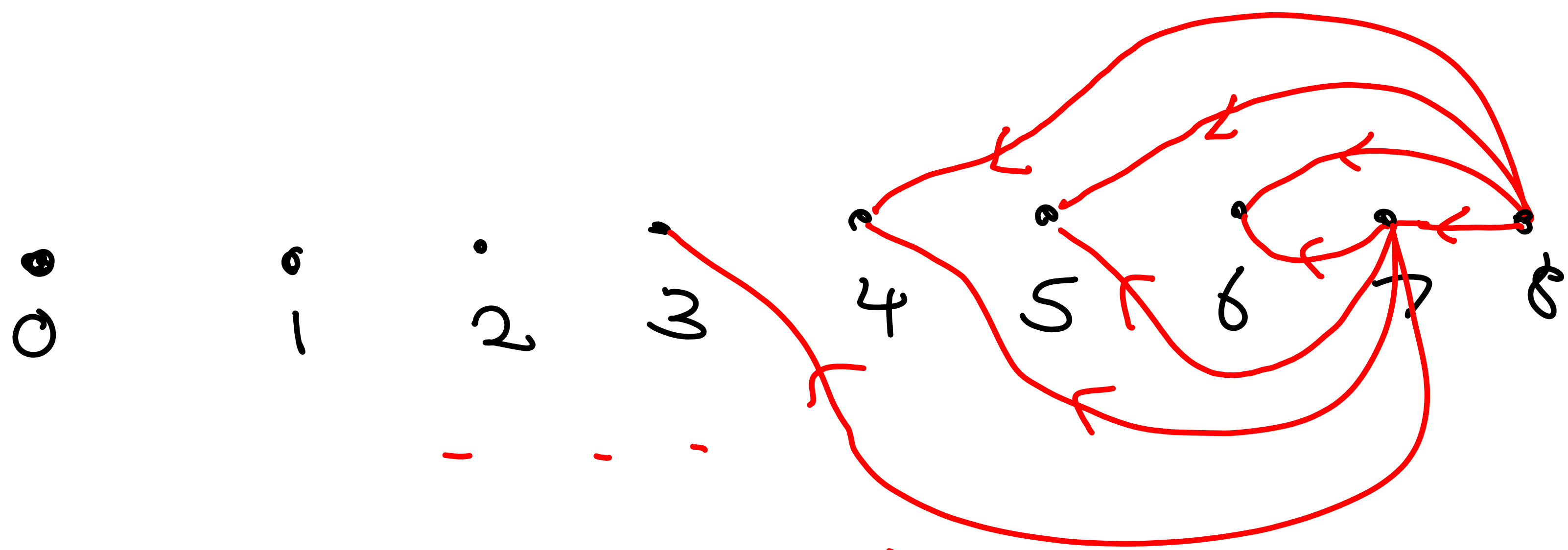
$X = \{\text{positions}\}$

$(x, y) \in A$ iff it is legal to move from position x to position y

Example 1

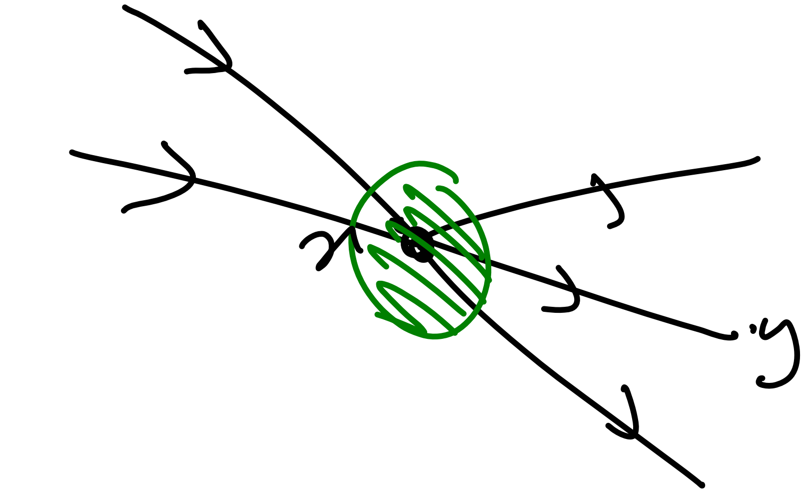


$\} \; n = 8$

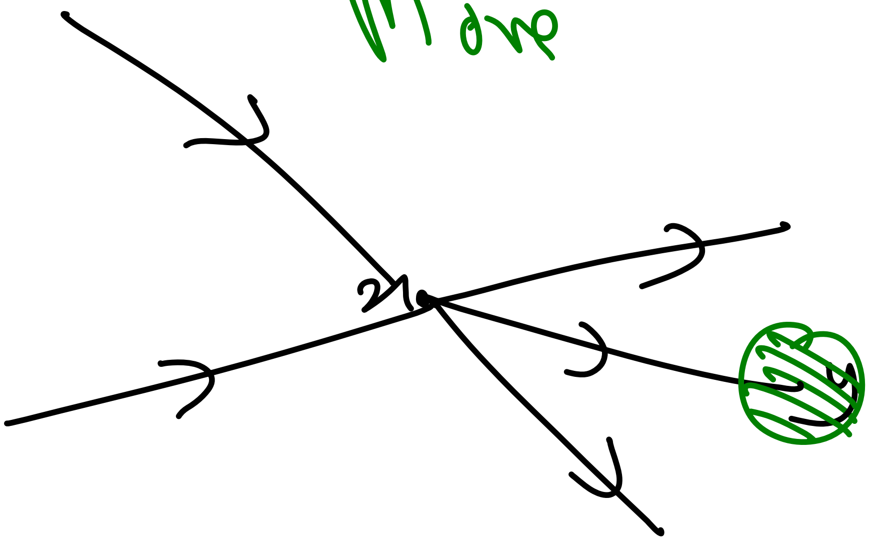


$$A = \{ (i, i+j) : 1 \leq j \leq 4 \}$$

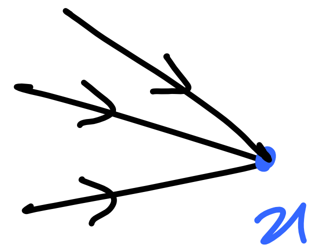
To play game: have a token on one vertex x say. To move push the token to y where $(x,y) \in A$



Move



Game terminates:



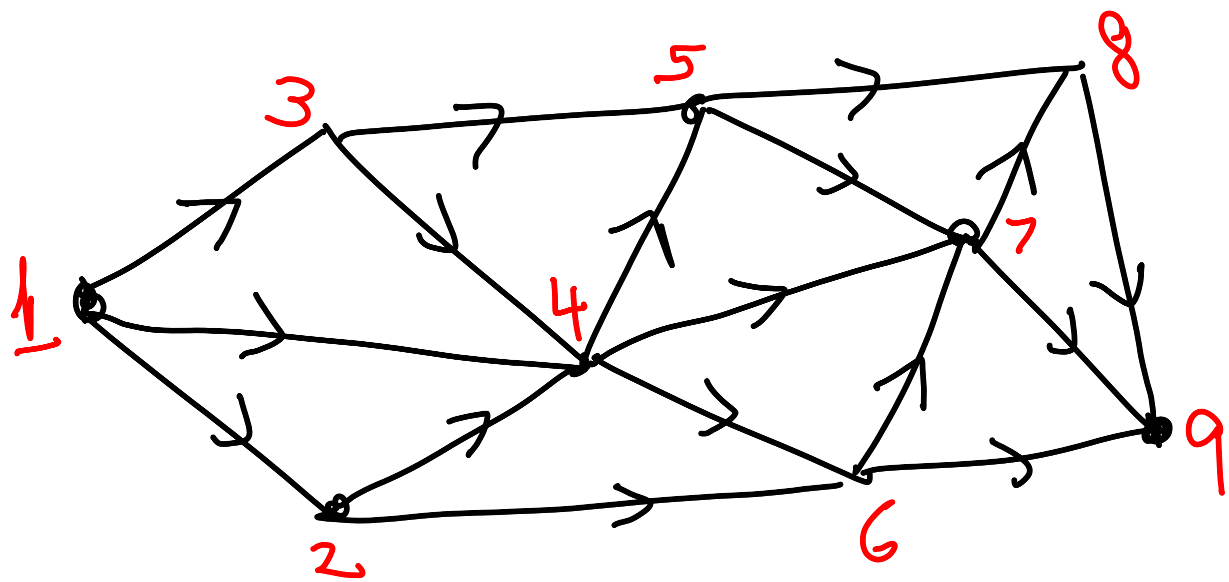
x is a "sink"

out-degree $= 0$.

Assumption: $|X| < \infty$ and no directed cycles DAG

A <u>topological numbering</u> of D is a function $f: X \to n = |X|$

$f(x) < f(y)$

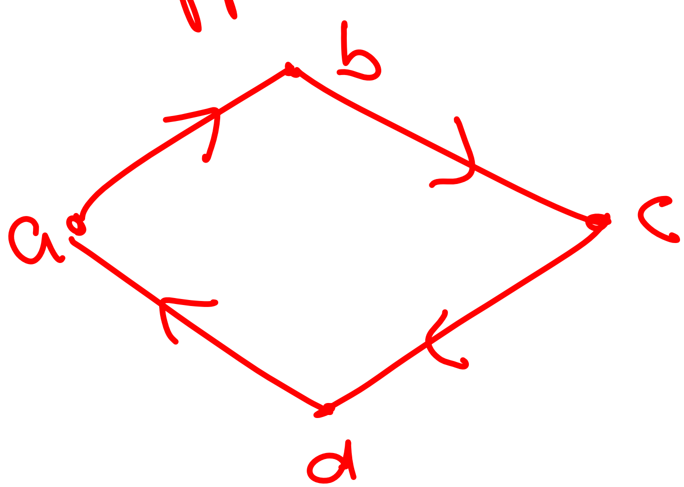


Claim

D is acyclic iff it has a topological numbering.

Proof

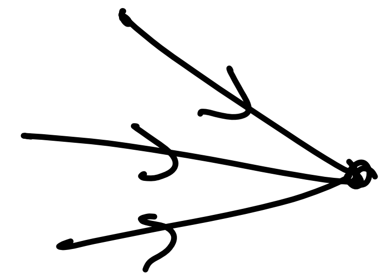
If: suppose $\exists$



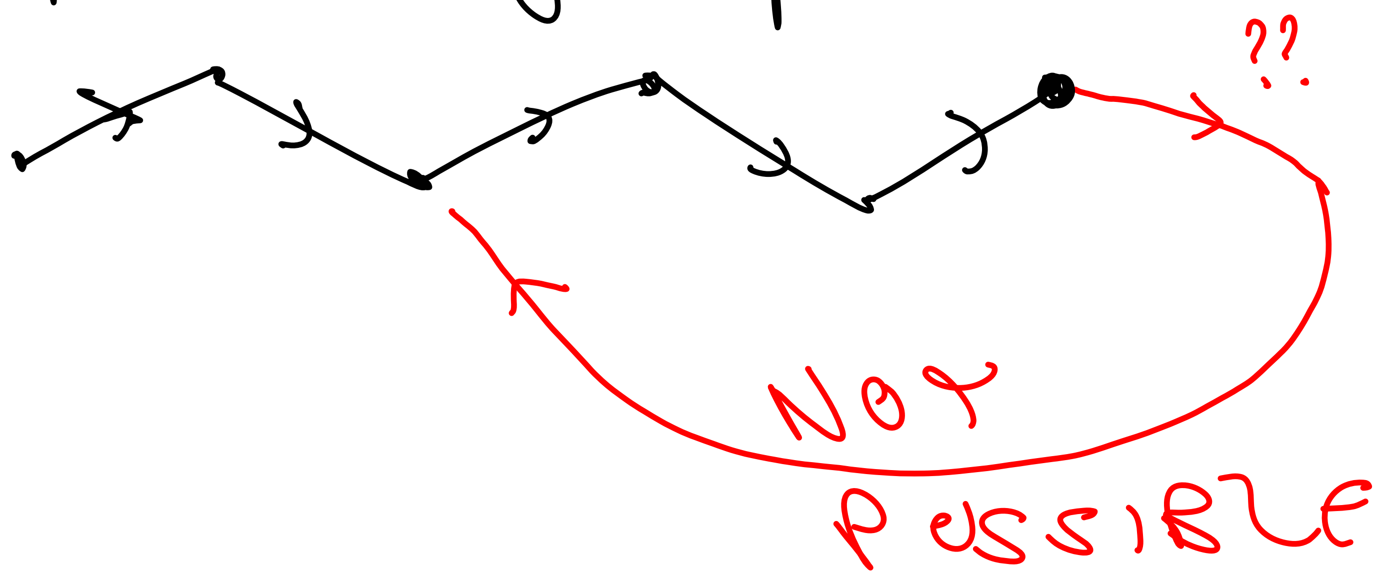
$f(a) < f(b) < f(c) < f(d) < f(a)$?

Only if: assume no cycles

There must be a sink.



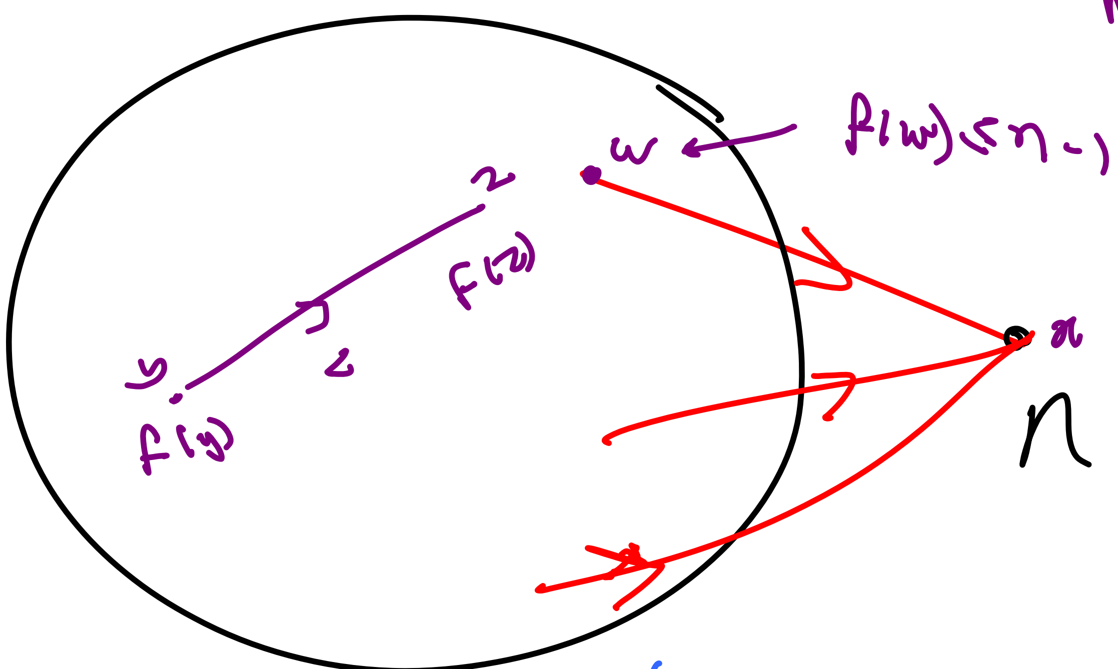
Let P be a longest path.



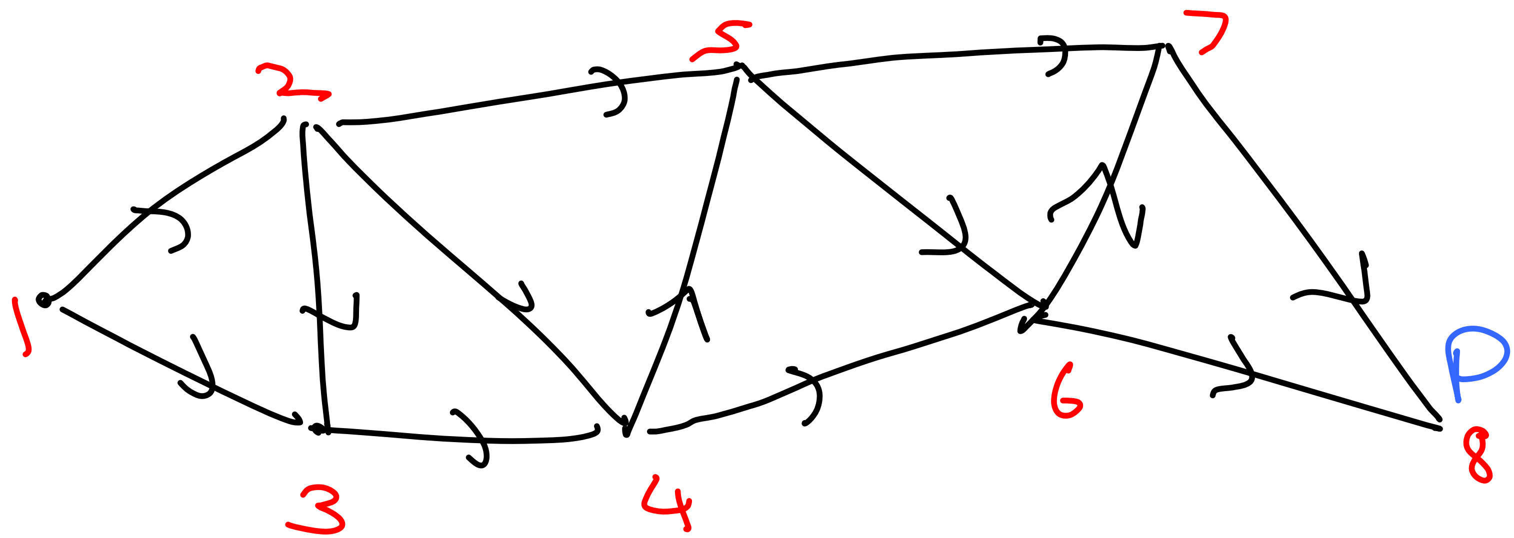
Ordering

Find a sink, x; $f(x) = n$

Give a topological numbering



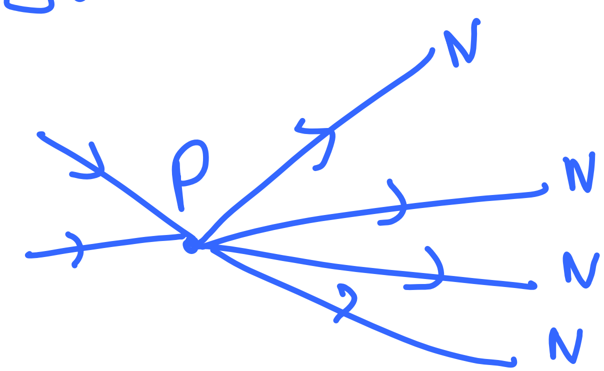
Use induction (recursion) to number remaining DAG $D - x$



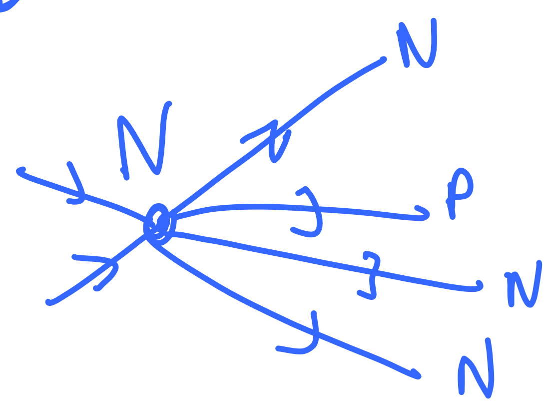
Label positions at N or P

1) We label in reverse topological order.

2) Label sink with P



or



The partition $X = N \cup P$

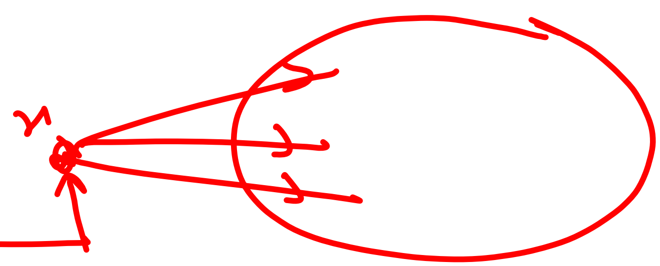
satisfies

$$x \in P \iff N^+(x) \subseteq N$$

$\exists$ a unique partition satisfying

Suppose (N, P) & (N', P') satisfy this

First place where N, N' differ



x sees the same thing in both partitions.

Sums of games

Games $G_1, G_2, \dots, G_p$

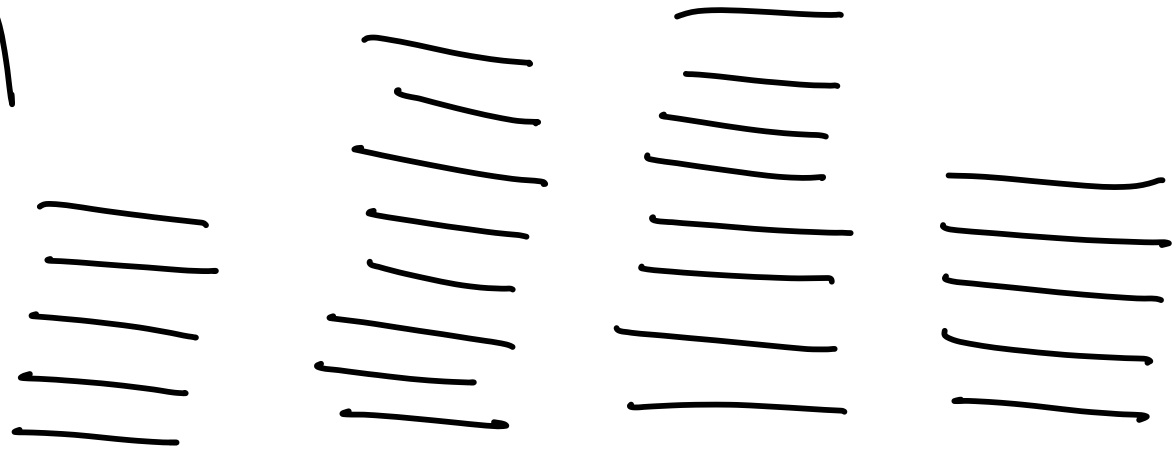
$$\begin{tikzcd}
G \arrow[dd] & = & G_1 \arrow[d] & \oplus & G_2 \arrow[d] & \oplus \cdots \oplus & G_p \arrow[d] \\
 & & (X_1, A_1) & & (X_2, A_2) & & (X_p, A_p) \\
D = (X_1 \times X_2 \times \cdots \times X_p) & & & & & &
\end{tikzcd}$$

$x_i' \in N_i^+(x_i)$

$$A = \{(x_1, x_2, \dots x_i, \dots x_p) \to (x_1, x_2, \dots x_i', \dots x_p)\}$$

Example

NIM



Move $\equiv$ take some chips from some non-empty pile.

G_n corresponds to a one pile game