11\04\09

Extremal Problems in set theory:

Typical question: suppose $\mathcal{A} \subseteq 2^{[n]}$ = { subsets of [n] } $\mathcal{P}_n$

and $\mathcal{A}$ has some property.

How big can $|\mathcal{A}|$ get?

Sperner Family.

$\mathcal{A} \subseteq P_n$ is a Sperner family if $A, B \in \mathcal{A}$ and $A \neq B$ then

$$A \not\subseteq B \;\&\; B \not\subseteq A$$

$\mathcal{A}$ is a collection of pair-wise incomparable sets.

Suppose

$$\mathcal{A}_k = \{ A \subseteq [n] : |A| = k \}$$

$$|\mathcal{A}_k| = \binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$$

So $\mathcal{A}_{\lfloor n/2 \rfloor}$ is a largest size Sperner family

Thm

If $\mathcal{A}$ is a Sperner family then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$

Proof

Follows from

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$$

LYM inequality

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \le 1$$

$$\Rightarrow \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \le 1$$

$$\Rightarrow |\mathcal{A}| \times \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \le 1$$

Proof of LYM inequality:

Let π be a random permutation of $1, 2, \ldots, n$

$$\mathcal{E}_A = \{ A = \{\pi(1), \pi(2), \ldots, \pi(|A|)\}\}$$

First $|A|$ elements of permutation coincide with A.

Suppose $\pi = 7, 4, 9, 40, 3, 5, \ldots$

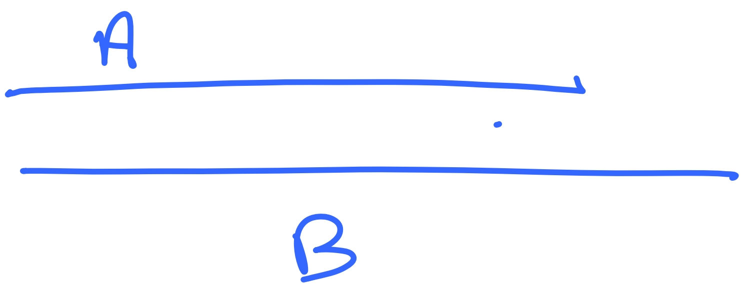
$\mathcal{E}_{\{4,7,9\}}$ occurs.

If $A, B \in \mathcal{A}$ then

$$\mathcal{E}_A \cap \mathcal{E}_B = \emptyset$$

Suppose $\neq \emptyset$ & $|A| \leq |B|$

$\pi(1), \pi(2), \ldots \ldots \pi(n)$



— contradiction.

The events are disjoint so

$$1 \geq \Pr\left(\bigcup_{A \in \mathcal{A}} \mathcal{E}_A\right)$$

$$= \sum_{A \in \mathcal{A}} \Pr(\mathcal{E}_A)$$

But $\Pr(\mathcal{E}_A) = \dfrac{1}{\binom{n}{|A|}}$.

Kraft's Inequality

$X_1, X_2, \ldots, X_m$ are a collection of sequences over an alphabet Σ of s

e.g.

$\Sigma = \{a, b, c\}$

$|X_i| = n_i$

$X_1 =$ aaabcc $\quad |X_1| = 6$

$X_2 =$ abc $\quad |X_2| = 3$

$X_3 =$ abababbabc $\quad |X_3| = 9$

$\vdots$

Suppose that no x_i is a prefix of an x_j

$x = a_1 a_2 \cdots a_k$ is a prefix of

$y = b_1 b_2 \cdots b_l$

if

$a_1 = b_1$

$a_2 = b_2$

$\vdots$

$a_k = b_k$

Kraft's Inequality:

$$\sum_{i=1}^{m} \frac{1}{s^{n_i}} \le 1$$

$n = \max\{n_1, n_2, \dots, n_m\}$

x is a random string of length n

$E_i = \{ x_i = \text{first } n_i \text{ symbols of } x \}$

$\langle \; x_i \; \rangle$

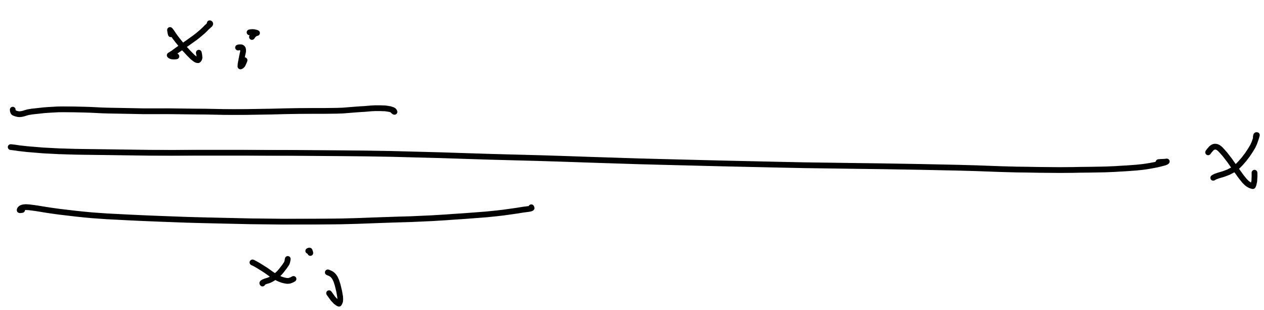
x

$$\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$$

because x_i is not a prefix of x_j

and vice-versa.

If $\neq \emptyset$



So

$$\underbrace{\Pr(\mathcal{E}_1)}_{1/s^{n_1}} + \underbrace{\Pr(\mathcal{E}_2)}_{1/s^{n_2}} + \cdots + \underbrace{\Pr(\mathcal{E}_m)}_{1/s^{n_m}} \le 1$$

Intersecting Families

$\mathcal{A}$ is an intersecting family

if $A, B \in \mathcal{A} \Rightarrow A \cap B \neq \emptyset$

$$|\mathcal{A}| \leq 2^{n-1}$$

Can't have A and $[n] \setminus A$

Partition $\mathcal{P}_n$ into 2^{n-1} pair $A, \bar{A}$

and can only have one from

$$\mathcal{A}_1 = \{ S \subseteq [n] : 1 \in S \}$$

$$|\mathcal{A}_1| = 2^{n-1}$$

Now suppose

$A \in \mathcal{A} \implies |A| = k$

1) $k \geq \frac{n}{2} \implies$ can have $|\mathcal{A}| = \binom{n}{k}$

2) $k \leq \lfloor \frac{n}{2} \rfloor$

Erdős-Ko-Rado: $|\mathcal{A}| \leq \binom{n-1}{k-1}$

Best possible, take all k-sets containing 1.