11\02\09

$$R(k,k) > 2^{k/2}$$

Show that if $n \le 2^{k/2}$ then $\exists$ a Red-Blue coloring of K_n without a mono k-clique.

Consider a random 2-coloring

$\mathcal{E}_R = \{\exists \text{ red } k\text{-clique}\}$

$\mathcal{E}_B = \{\exists \text{ blue } k\text{-clique}\}$

To show:

$$\Pr(\mathcal{E}_R \cup \mathcal{E}_B) < 1$$

Let $C_1, C_2, \cdots, C_N$, $N = \binom{n}{k}$

be an enumeration of the

k-cliques of K_n.

$$\mathcal{E}_{R,j} = \{ \text{Clique } C_j \text{ is Red} \}$$

$$\mathcal{E}_R = \bigcup_{i=1}^{N} \mathcal{E}_{R,i}$$

$$
\begin{aligned}
\Pr(\mathcal{E}_R \cup \mathcal{E}_B) &\leq \Pr(\mathcal{E}_R) + \Pr(\mathcal{E}_B) \\
&= 2\Pr(\mathcal{E}_R) \\
&= 2\Pr\left(\bigcup_{i=1}^{N} \mathcal{E}_{R,i}\right) \\
&\leq 2\sum_{i=1}^{N} \Pr(\mathcal{E}_{R,i}) \\
&= 2\sum_{i=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}}
\end{aligned}
$$

$$\Pr(\mathcal{E}_R \cup \mathcal{E}_B) \leq 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$< 2\,\frac{n^k}{k!}\left(\frac{1}{2}\right)$$

$$\boxed{n \leq 2^{k/2}}$$

$$\leq \frac{2^{1+\frac{k^2}{2}-\binom{k}{2}}}{k!}$$

$$= \frac{2^{1+k/2}}{k!} \leq 1 \,, \quad k \geq 3 .$$

Ramsey's Theorem: General Case.

all sets of size 2 in [n]

Graph edges — 2 colors

$\Downarrow$ large n

Set of size k all of whose 2-sets are same color

all colorings of sets of size $r \geq 2$ in [n] — s colors

$\Downarrow$

$n \geq N(q_1, q_2, \ldots, q_s : r)$

$\exists$ an $1 \leq i \leq s$ and a set of size q_i all of whose r-tuples are colored i.

Schur's Theorem

$$r_k = N(\underbrace{3, 3, \ldots, 3}_{k} : 2)$$

Thm
If $S_1, S_2, \ldots, S_k$ partition $[r_k]$
then $\exists\, i$ and $x, y, z \in S_i$ such
that

$$x + y = z$$

Take the partition (coloring)

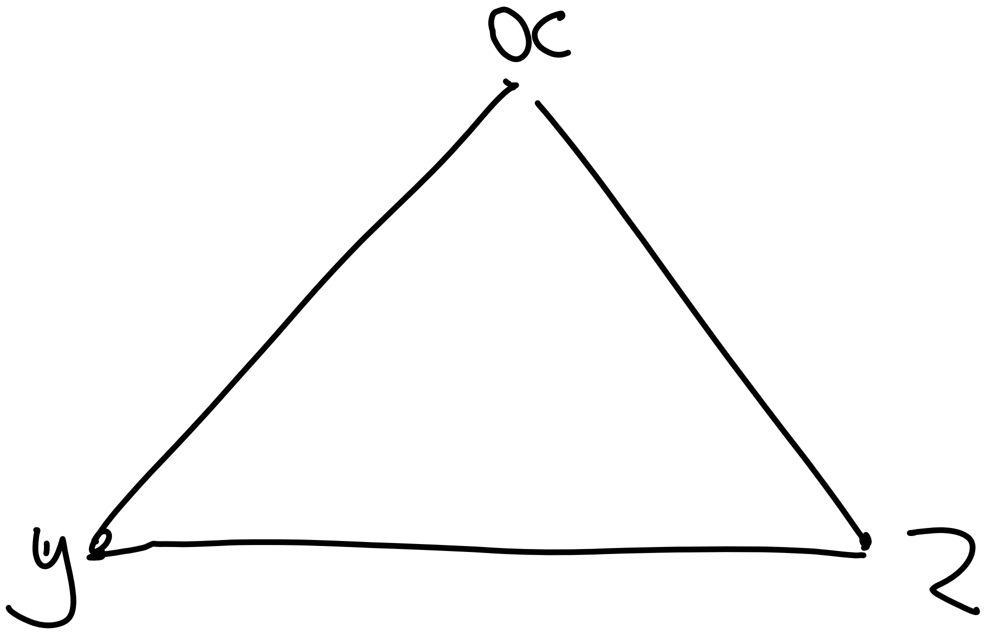
$S_1, S_2, \ldots, S_k$.

Use it to color K_{r_k}

Edge (u,v) is given the color j

$|u-v| \;-\; |u-v| \in S_j \Rightarrow$ color (u,v) with j

$\exists$ a Δ of one color

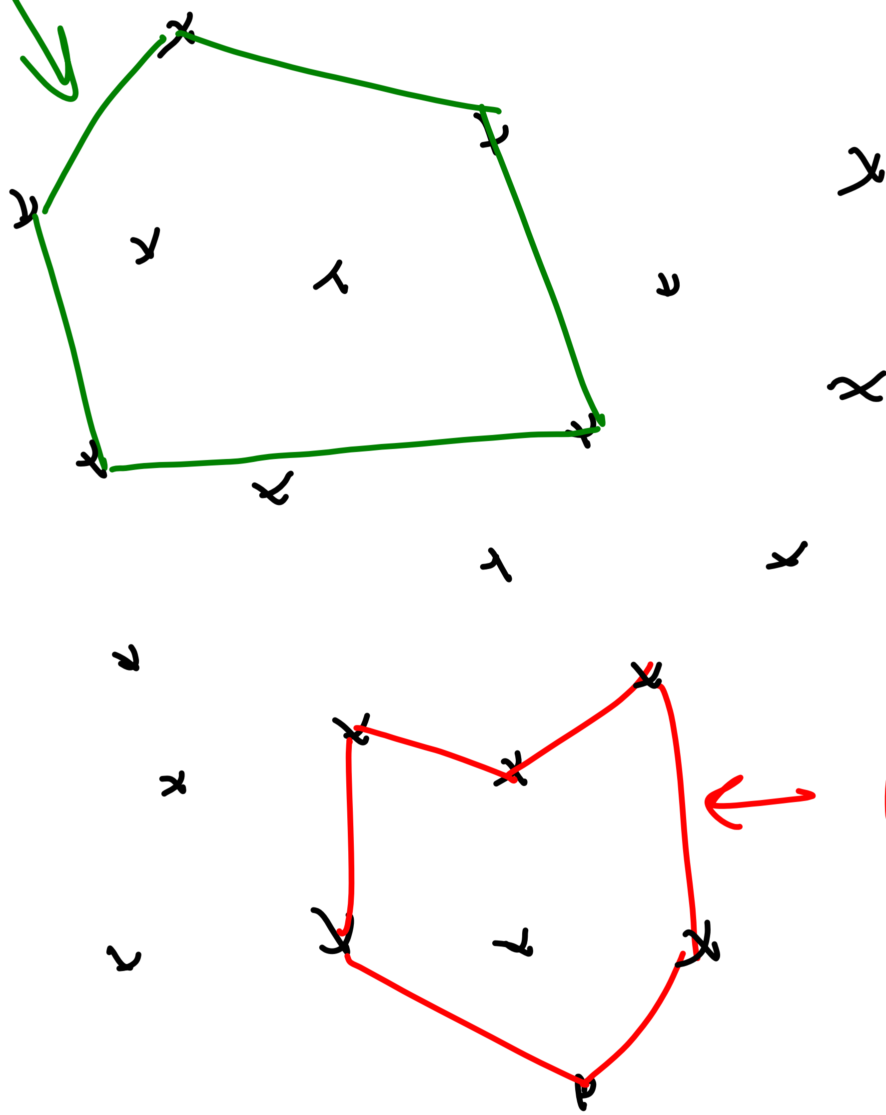


$$x < y < z$$

$$\begin{array}{r}
y - x \\
+\; z - y \\
=\; z - x
\end{array}$$

all have same color.

convex polygon



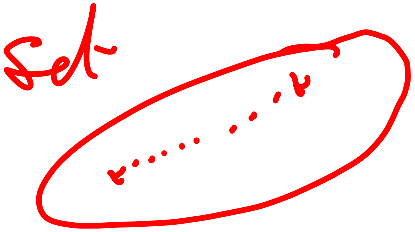
n points

$$n \geq N(k,k;3)$$

2-color triples

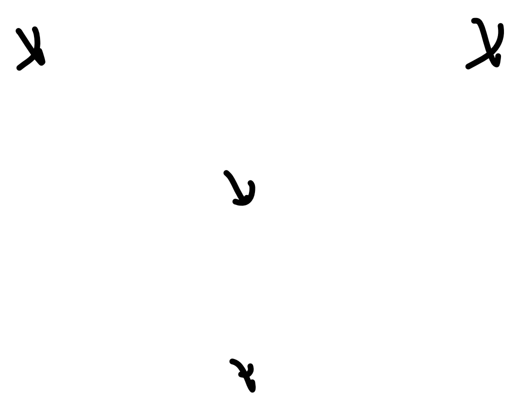
non-convex

convex set



If $N \geq 3$ then $\exists$ convex polygon of size 3. $\rightarrow$ trivial

$N \geq 4$, may not get convex polygon of size 4

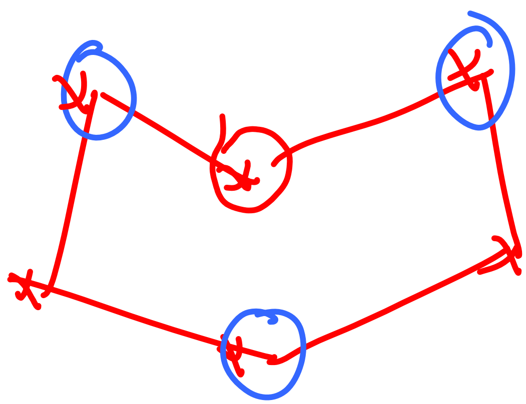


$$n \geq N(k, k : 3)$$
$$\Downarrow$$
$\exists$ convex k-gon.

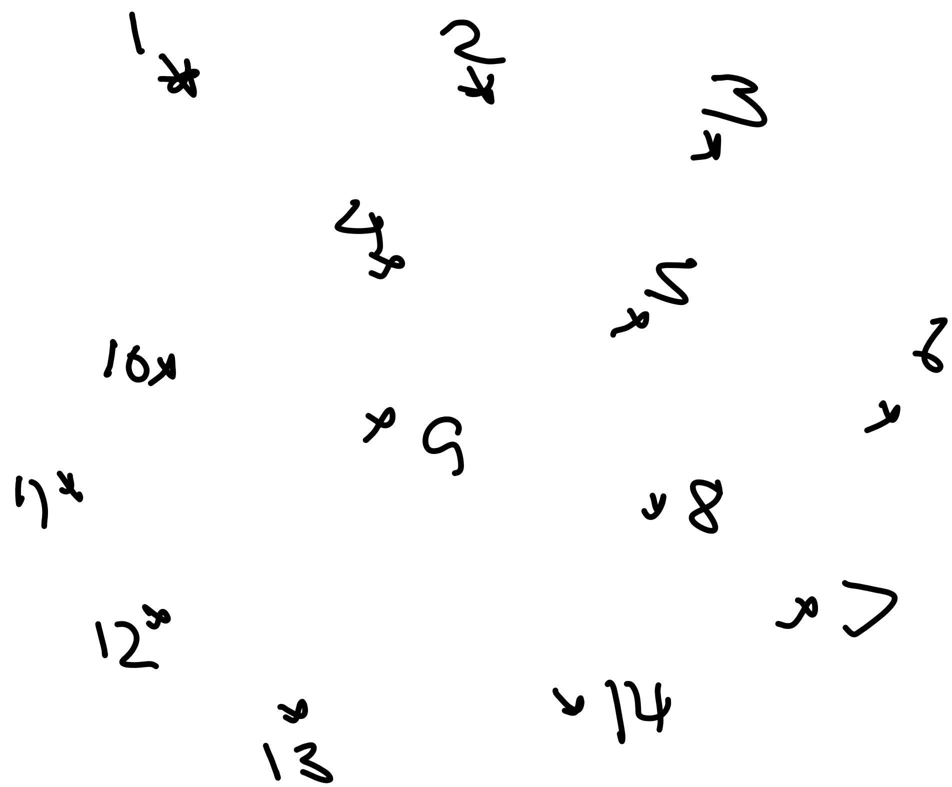
Observation

If every 4-subset of $Y \subseteq X$ is convex, then so is Y.

Suppose Y is not convex



choose Δ containing $\circledast$



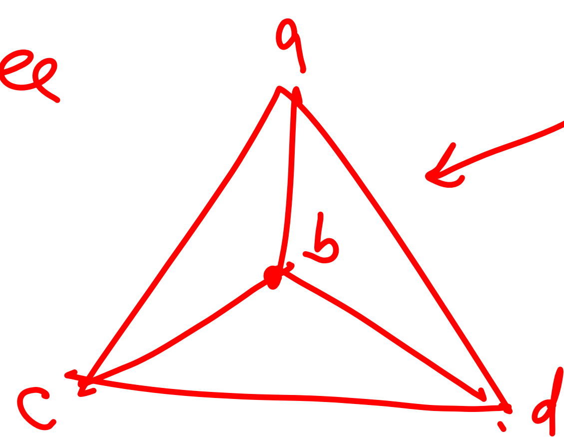
Color the $\binom{14}{3}$ triangle $\quad i < j < k$

Applying Ramsey's Theorem.

$\exists k$ set, K all of whose triples have same color.

Any 4 pts from K are convex

You don't see



all colored Red.

For example

$a < b < c < d$