10\28\09

Pigeon Hole Principle

positive integers

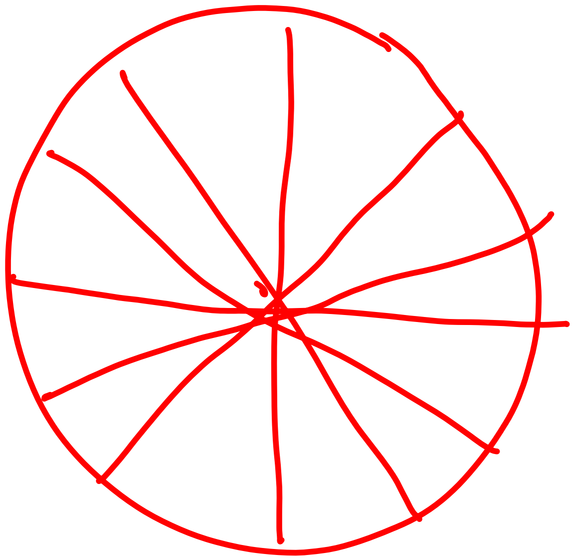
$$x_1 + x_2 + \cdots + x_n \geq m$$

$$\Rightarrow \exists\, i : x_i \geq \left\lceil \frac{m}{n} \right\rceil$$

$$f : [m] \longrightarrow [n] \Rightarrow \exists\, i \in [n]$$

$$\text{s.t.} \quad x_i = |f^{-1}(i)| \geq \left\lceil \frac{m}{n} \right\rceil.$$

DISK 1

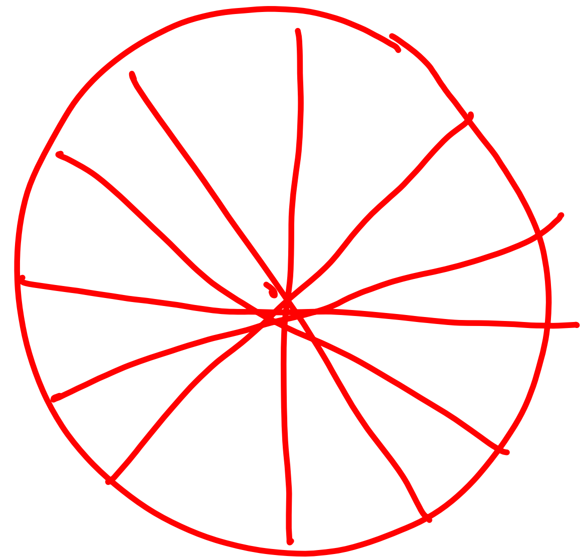


200 Sectors

100 R

100 B

DISK2



200 Sectors

Arbitrarty colored R&B.

CLAIM: ONE CAN PLACE DISK 2 ONTO DISK 1 SO THAT ≥ 100 SECTORS HAVE SAME COLOR.

200 ways of placing Disk 1 onto Disk 2.

q_i = # matches for position i

$$\overbrace{q_1+q_2+\cdots+q_{200}}^{\text{by rows}} = \overbrace{200\times 100}^{\text{by columns}} \qquad (**)$$

$$\Rightarrow \exists i : q_i \geq 100$$

$$\begin{array}{ccc} & \text{sector } j & \\ \text{position } i & \left[\begin{array}{ccc} \quad & 1_{\substack{\text{color}\\ \text{match}}} & \quad \end{array}\right] & \leftarrow q_i \ 1\text{'s} \\ & \text{100 1's in each column} & \end{array}$$

Alternate explanation:

Place Disk 2 randomly on Disk 1.

$$X_i = \begin{cases} 1 & \text{if sector matches} \\ 0 & \text{otherwise} \end{cases}$$

$$X = X_1 + \cdots + X_{200} = \#\text{matches}$$

$$E(X) = \underbrace{E(X_1)}_{\frac{1}{2}} + \cdots + \underbrace{E(X_{200})}_{\frac{1}{2}} = 100$$

$\Rightarrow \exists$ placement with ≥ 100 matches.

Erdős–Szekerés.

$a_1, a_2, \ldots a_{k^2+1}$ are reals.

Then $\exists$ a monotone subsequence of length $k+1$

$i_1 < i_2 < \cdots < i_l$ is monotone if

$$a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_l}$$

or

$$a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_l}$$

increasing

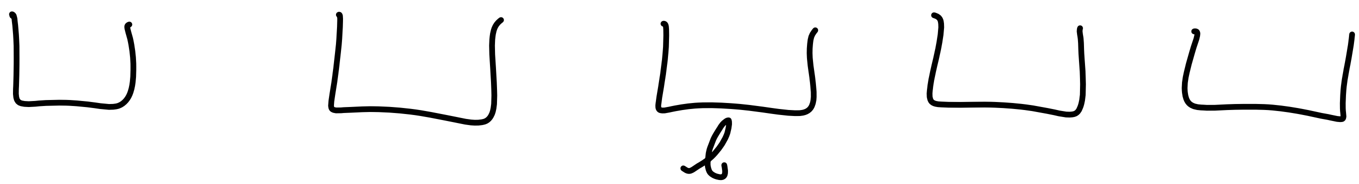
x x x

3 8 4 7 33 12

* *

decreasing

k pigeonholes.



$\ell(i)$ = length of longest monotone increasing subsequence starting at a_i

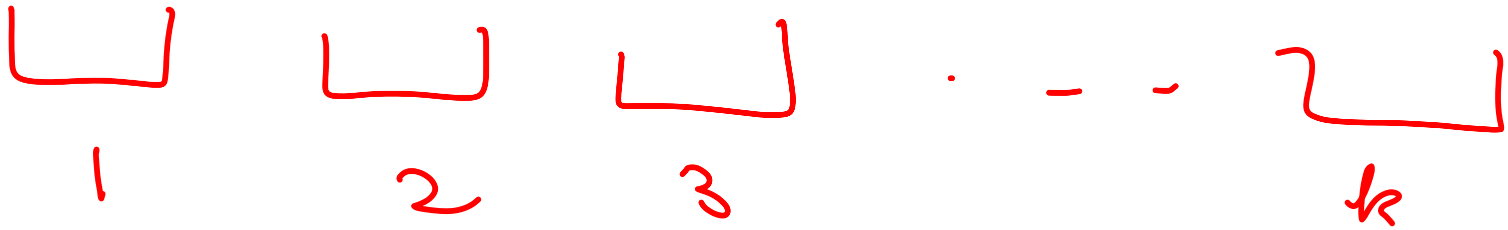
7 4 8 3 4 3 6 9 4 2

$\ell(1) = 3$ $\ell(3) = 2$

$\ell(2) = 4$ . . . .

Case 1: $\exists\, i$ s.t. $\ell(i) \geq k+1$

Case 2: $\ell(i) \leq k, \ \forall i$



Put i into bucket $\ell(i)$

k^2+1 numbers $\longrightarrow$ k buckets

$\exists$ bucket with $\geq k+1$ indices

$$\boxed{i_1 < i_2 < \cdots < i_{k+1}}$$

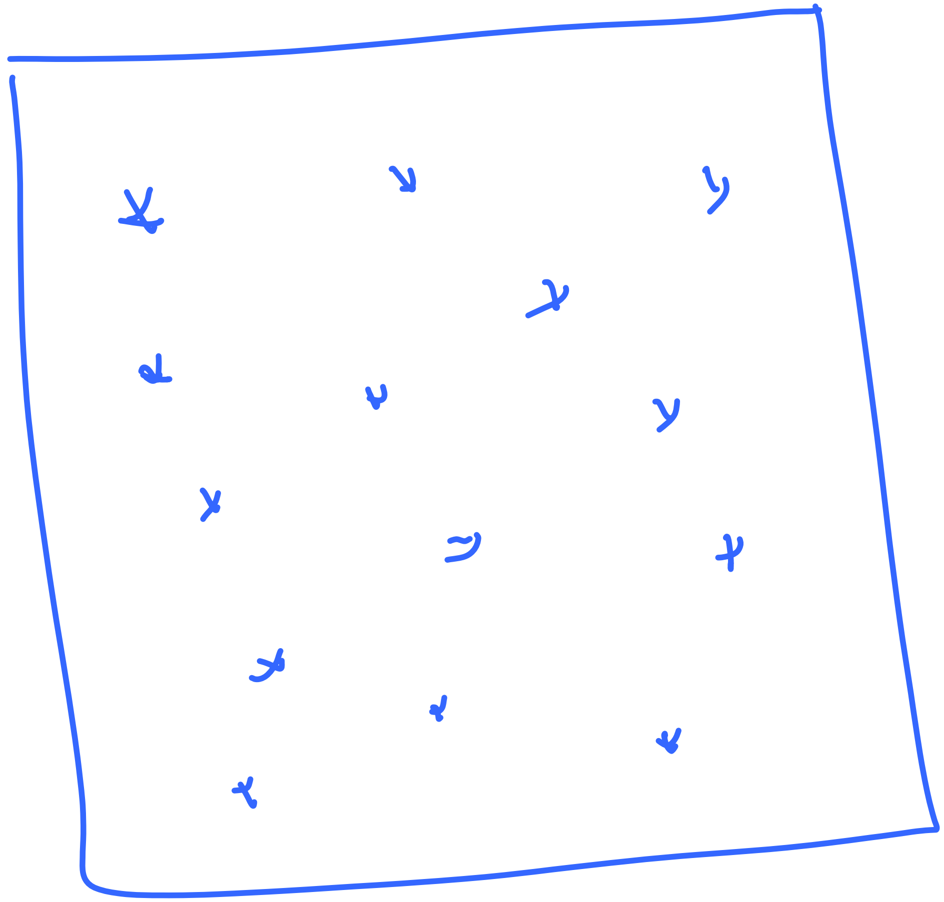
big bucket

Claim: $a_{i_1} > a_{i_2} > \cdots > a_{i_{k+1}}$

Suppose $a_{i_t} \leq \boxed{a_{i_{t+1}} \leq a_{j_2} \leq a_{j_3} \leq \cdots \leq a_{j_\ell}}$

$\leftarrow \exists$ mono ~~tone~~ increasing sequence of length ℓ.

$\Rightarrow \ell(a_{i_t}) \geq \ell + 1$ — contradiction



n points $\implies \exists\, \Delta$ of small area $\sim O\left(\frac{1}{n}\right)$.