21-301 Combinatorics
Homework 3
Due: Monday, September 29

1. Show that the number of sequences out of $\{a, b, c\}^n$ which do not contain a consecutive sub-sequence of the form xx where $x = a, b$ satisfies the recurrence $b_0 = 1, b_1 = 3$ and

$$b_n = b_{n-1} + 2(b_{n-2} + \cdots + b_0) + 2b_0.$$

   [Hint: Consider the number of sequences where the first c from the left is at position k.]

   Deduce from this that

$$b_n = 2b_{n-1} + b_{n-2}.$$

2. Show that the number of sequences out of $\{a, b, c\}^n$ which do not contain a consecutive sub-sequence of the form abc satisfies the recurrence $b_0 = 1, b_1 = 3, b_2 = 9$ and

$$\begin{aligned} b_n &= 2b_{n-1} + c_n && (1) \\ c_n &= c_{n-1} + b_{n-2} + c_{n-2} + b_{n-3} && (2) \end{aligned}$$

   where c_n is the number of such sequences that start with a.

   Now find a recurrence only involving b_n, by using (1) to eliminate c_n from (2).

3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the recurrence relation

$$a_n + 3a_{n-1} + 2a_{n-2} = n \quad \text{for } n \geq 2$$

   with initial conditions $a_0 = 1$ and $a_1 = 3$. Determine the generating function for this sequence, and use the generating function to determine a_n for all n.