10\3\08

Independence.

Events A & B are independent

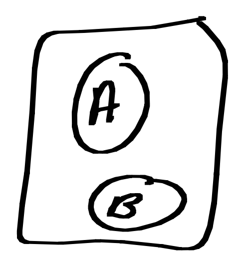
if $$P(A \cap B) = P(A)\,P(B)$$

or

$$P(A|B) = P(A)$$

Not the same as $A \cap B = \emptyset$

i.e. don't confuse indep. & disjoint



Independence from disjoint sets of bits.

$$\Omega = \{0,1\}^n = \{x_1 x_2 \cdots x_n\}$$

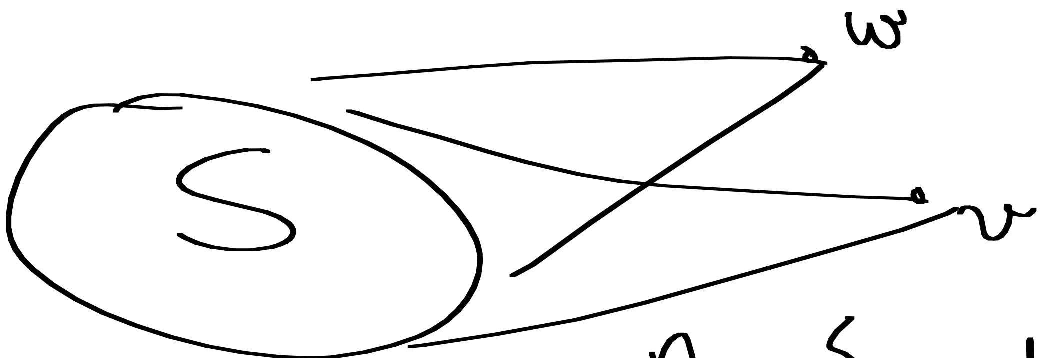
A depends first 3 bits e.g. $A = \{x_1 + x_2 + x_3 \geq 2\}$

B depends on bits 4 – 8

e.g. $B = \{x_4 = \cdots = x_8 = 0\}$

A and B are independent

Tournament Problem



$|S| = k$

$A_v = \{ v \text{ beats } S \}$

A_v & A_w are independent if $v \neq w$

Random Variables

A function $\zeta : \Omega \to R$
zeta

is called a random variable.

$\Omega = [6]^2$ — Rolling 2 dice

$p_k = P_r(\zeta = k)$ $\quad \zeta = x_1 + x_2$

Gives new prob. space on integers

$$P_r(\zeta = 4) = P((1,3) \text{ or } (2,2) \text{ or } (3,1))$$
$$= 3/36$$

$\Omega = \{$ colorings, m balls
n colors $\}$ $x_1 + \cdots + x_n = m$

$\zeta =$ # of colors actually used.

$$P_k = \frac{\text{\# colorings using } k \text{ colors}}{\text{\# colorings altogether}}$$

$$= \frac{\binom{n}{k}\binom{m-1}{k-1}}{\binom{n+m-1}{m}}.$$

Binomial $B_{n,p}$

Toss n coins.

$Pr(H) = p$

$B_{n,p} =$ # Heads

$$Pr(B_{n,p} = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Expectation (average)

Z is a random variable

$$E(Z) = \sum_{\omega} Z(\omega)\, P(\omega)$$

$$= \sum_{k} k\, p_k$$

$Z = x_1 + x_2$

$$E(Z) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + \dots 12 \times \frac{1}{36} = ?$$

Z = # colors used when m balls are randomly colored with n colors.

$$k\binom{n}{k} = \frac{n!}{(k-1)!\,(n-k)!} = n\binom{n-1}{k-1}$$

$$E[Z] = \sum_{k=1}^{n} k \frac{\binom{n}{k}\binom{m-1}{k-1}}{\binom{n+m-1}{m}}$$

$$= n\sum_{k=1}^{n} \frac{\binom{n-1}{k-1}\binom{m-1}{k-1}}{\binom{n+m-1}{m}}$$

$$= n \sum_{k=1}^{n} \frac{\binom{n-1}{k-1}\binom{m-1}{m-k}}{\binom{n+m-1}{m}}$$

$$= n \sum_{k=1}^{n} \frac{\binom{n-1}{k-1}\binom{m-1}{m-k}}{\binom{n+m-1}{m}}$$

$$= n \frac{\binom{n+m-2}{m-1}}{\binom{n+m-1}{m}}$$

$$= \frac{mn}{n+m-1} .$$

Suppose X, Y are random variable on Ω then

$$E(X+Y)$$

$$= E(X) + E(Y)$$

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n)$$

$$E(B_{n,p})$$

indicator variables

$$B_{n,p} = X_1 + X_2 + \cdots + X_n$$

$$X_i = 0 \text{ or } 1$$

$$Pr(X_i = 1) = p$$

$$E(B_{n,p}) = E(X_1) + E(X_2) + \cdots + E(X_N)$$

$$0 \times (1-p) + 1 \times p = p$$

$$= np$$

Suppose $\Omega = \{H, T\}^n$

$Z = \#$ of times we see $HTTH$

Outcome in $\quad HHTTHTTHHHTTH$

$\# = 3$

$$E(Z) = E(X_1) + E(X_2) + \dots + E(X_{n-3})$$

$X_i = 1$ iff $X_i X_{i+1} X_{i+2} X_{i+3} = HTTH$

$$P[X_1 = 1] = p^2(1-p)^2$$

$$E(Z) = (n-3)p^2(1-p)^2$$

m balls, n colors

Z = # colors used

$= Z_1 + Z_2 + \cdots + Z_n$

$Z_i = 1 \iff i$ is used.

$$E(Z) = n\,E(Z_1)$$
$$= n \Pr(Z_1 \neq 0)$$
$$= n\,(1 - \Pr(Z_1 = 0))$$

$$\Pr(Z_1 = 0) = \frac{\#\text{ colorings using } 2, 3, \ldots n}{\text{total } \#\text{ colorings}}$$

$$= \frac{\binom{n+m-2}{m}}{\binom{n+m-1}{m}}$$

$$= \frac{n-1}{n+m-1}$$

$$E(Z) = \frac{mn}{n+m-1}$$