10\3\08

$R(k,k) < 4^k$

Lower bound.

$$R(k,k) > 2^{k/2}$$

Show that if $n \le 2^{k/2}$ then $\exists$ a 2-coloring of the edges of K_n s.t. $\nexists$ Red k-clique or a Blue k-clique.

Random coloring will suffice.

We randomly color edges of K_n, Red\ Blue

$$\mathcal{E}_R := \{\exists \text{ Red } k\text{-clique}\}$$

$$\mathcal{E}_B := \{\exists \text{ Blue } k\text{-clique}\}$$

We show

$$\Pr(\mathcal{E}_R \cup \mathcal{E}_B) < 1$$

$$\leq \Pr(\mathcal{E}_R) + \Pr(\mathcal{E}_B) = 2\Pr(\mathcal{E}_R)$$

$$\Pr(\mathcal{E}_R \cup \mathcal{E}_B) \leq 2\Pr(\mathcal{E}_R)$$

$C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ is the list of k-cliques of K_n.

$$\mathcal{E}_{R,j} \equiv \{ C_j \text{ is Red} \}$$

$$\mathcal{E}_R = \bigcup_{j=1}^{N} \mathcal{E}_{R,j}$$

$$
\begin{aligned}
\Pr(\mathcal{E}_R \cup \mathcal{E}_B) &\le 2\Pr(\mathcal{E}_R) \\
&\le 2\sum_{i=1}^{N} \Pr(\mathcal{E}_{R,i}) \\
&= 2\sum_{i=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&< 2\,\frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&< \frac{2^{1+k^2/2-\binom{k}{2}}}{k!} \\
&= \frac{2^{k/2+1}}{k!} < 1, \quad k \ge 3.
\end{aligned}
$$

$n \le 2^{k/2}$

Ramsey's Theorem in General

So far we

$\cancel{2}^{\,s}$ - color ← more colors

$\cancel{2}^{\,r}$ - subsets of $[n]$

Hypergraphs – edges are r-sets

Ramsey's Theorem: Given, $r, s, q_1, \dots, q_s$ ∃

$\exists\ N(q_1, q_2, \dots, q_s; r)$ such that if $n \geq N$

in <u>every</u> s coloring of $\binom{[n]}{r}$, ∃ i and

a set S, $|S| = q_i$ and $\binom{S}{r}$ is i-colored.

Assume first that $s=2$.

<u>N exists</u>

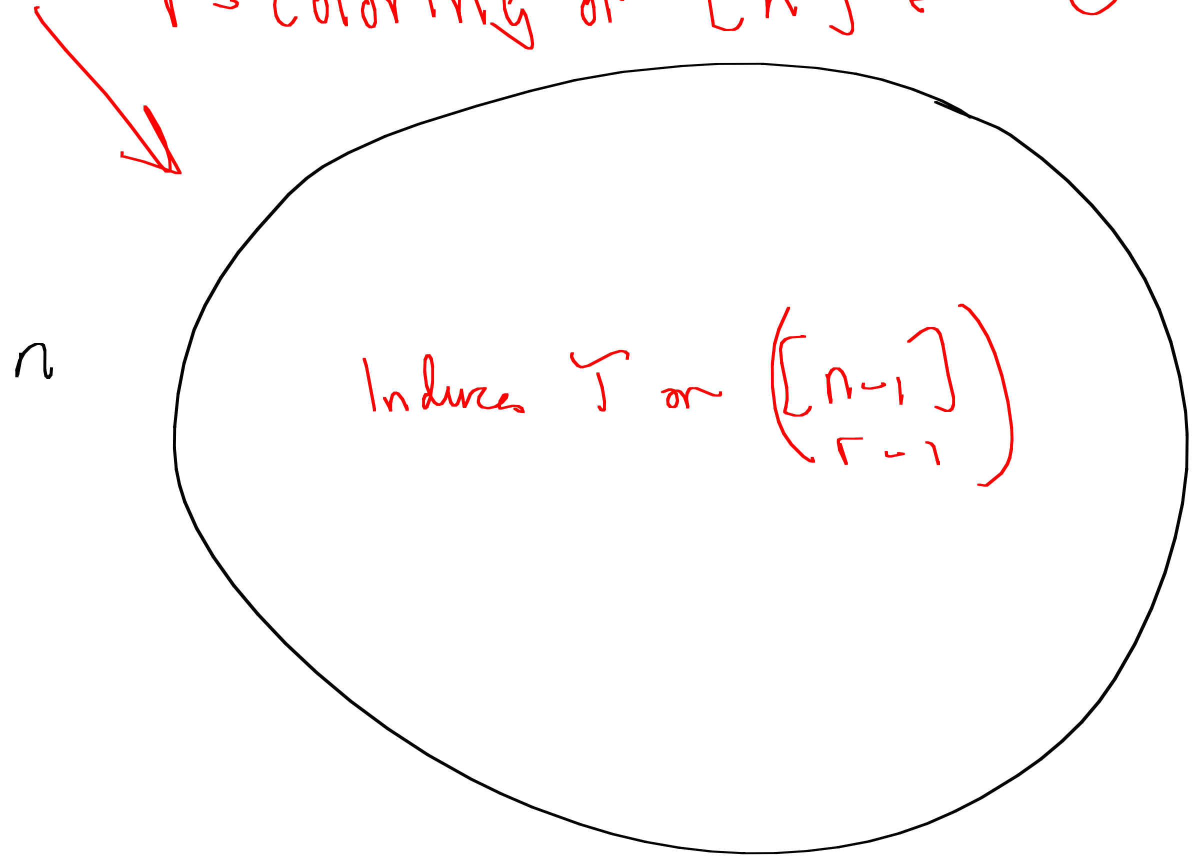
Induction on r.

True for $r=2$.

Assume true for r and $\underbrace{p'+q'<p+q}$

Given r, induction on $p+q$.

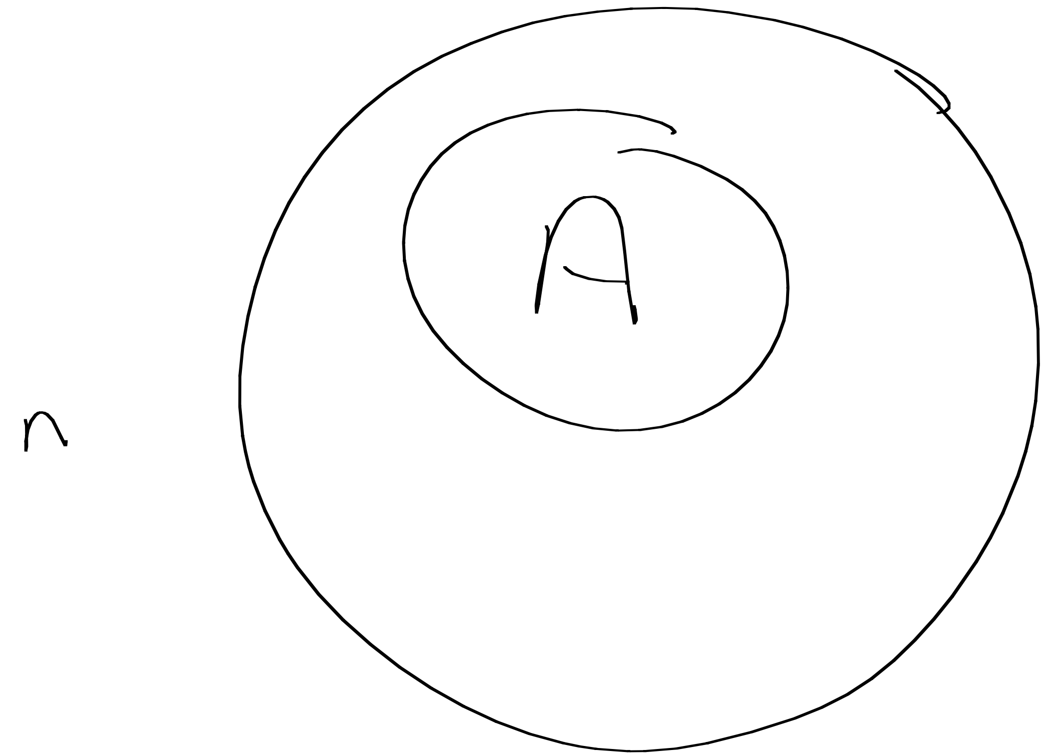
r-coloring of $[n] \leftarrow \sigma$



Represent $[n]$

σ induces an 2-coloring of $\binom{[n-1]}{r-1} \leftarrow \tau$

$$\tau[S] = \sigma[S+n]$$



$\binom{A}{r}$ is either all Red or all Blue

A itself is really big — we look at σ on A

we ask for $p-1, q$

2 colors

4 colors

Light Red, Dark Red, Light Blue, Dark Blue

4 color edge