1. Let \( s \geq 1 \) be fixed. Let \( \mathcal{A} \) be a family of subsets of \([n]\) such that there do not exist distinct \( A_1, A_2, \ldots, A_{s+1} \) such that \( A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{s+1} \). Show that

\[
\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq s.
\]

**Solution:** Let \( \pi \) be a random permutation of \([n]\). Let \( \mathcal{E}(A) \) be the event \( \{\pi(1), \pi(2), \ldots, \pi(|A|) = A\}\). Let

\[
Z_i = \begin{cases} 1 & \text{if } \mathcal{E}(A_i) \text{ occurs.} \\ 0 & \text{otherwise.} \end{cases}
\]

and let \( Z = \sum_i Z_i \) be the number of events \( \mathcal{E}(A_i) \) that occur. Now our family is such that \( Z \leq s \) for all \( \pi \) and so

\[
\mathbb{E}(Z) = \sum_i \mathbb{E}(Z_i) = \sum_i \Pr(\mathcal{E}(A_i)) \leq s.
\]

On the other hand, \( A \in \mathcal{A} \) implies that \( \Pr(\mathcal{E}(A)) = \frac{1}{\binom{n}{|A|}} \) and the required inequality follows.

2. Let \( G = (V, E) \) be an \( r \)-regular graph with \( n \) vertices. \( S \subseteq V \) is a dominating set if \( w \notin S \) implies that there exists \( v \in S \) for which \( \{v, w\} \in E \). Show, by the probabilistic method, that \( G \) has a dominating set of size at most \( \frac{1 + \ln r}{r} n \).

**Solution:** Let \( p = \frac{\ln r}{r} \) and let \( S_1 \) be a random sub-set of \( V \) where each \( v \in V \) is placed into \( S \), independently with probability \( p \). Let \( S_2 \) be the set of vertices that are not adjacent to any vertex of \( S_1 \). The set \( S = S_1 \cup S_2 \) is a dominating set.

\[
\mathbb{E}(|S|) = \mathbb{E}(|S_1|) + \mathbb{E}(|S_2|) = np + n(1 - p)^{r+1} \leq np + ne^{-rp} \leq \frac{1 + \ln r}{r} n.
\]

So there must be a dominating set of the required size.
3. Let $G = (V, E)$ be a graph with $kn$ vertices. Show, by the probabilistic method, that there is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ with $|V_i| = n$, $i = 1, 2, \ldots, k$ such that at most $|E|/k$ of the edges of $G$ have both of their endpoints in the same part of the partition.

**Solution:** Let $V_1, V_2, \ldots, V_k$ be a random partition of the vertex set. Let $e = (v, w)$ be an edge of $E$. Then

$$\Pr(\exists i : e \subseteq V_i) = \sum_{i=1}^{k} \Pr(w \in V_i \mid v \in V_i) \Pr(v \in V_i) =$$

$$\sum_{i=1}^{k} \frac{(kn-2)}{(kn-1)} \frac{1}{k} = \frac{n-1}{kn-1} \leq \frac{1}{k}.$$ 

If $Z$ is the number of edges of $G$ that have both of their endpoints in the same part of the partition, then $\mathbb{E}(Z) \leq |E|/k$ and so the required partition must exist.