Chains: $a_1 < a_2 < \ldots < a_p$

Anti-Chains: $b_1, b_2, \ldots, b_q$ are pairwise incomparable.

$C$ is a chain

$A$ is an anti-chain $\implies |C \cap A| \leq 1$

covered by chains & anti-chains.
Covers by anti-chains

Suppose $A_1, A_2, \ldots, A_m$ are anti-chains.

$C = \{a_1, a_2, \ldots, a_p\}$ is a chain.

Then $M \geq p$

Smallest $M = \text{largest } p$

Proof: by induction on length of longest chain. $M$
Base Case: $\mu = 1 \Rightarrow P$ is an anti-chain.

\[ A = \{ \text{maximal elements} \} \]

$x$ is maximal $\forall y: y > x$

$P' = P / A :$ $P'$ does not have a chain of length $\mu$

$P' = \bigcup_{i=1}^{\mu-1} A_i$ \text{ anti-chain}

$P = A_1 \cup \ldots \cup A_{\mu-1} \cup A_\mu$

$a_1 < a_2 < \ldots < a_\mu < y \in A$

Contradiction.
Covers by chains

Suppose \( C_1, C_2, \ldots, C_m \) are chains.

\[ A = \{ a_1, a_2, \ldots, a_p \} \]

is an anti-chain.

Then \( m \geq p \)

Smallest \( M = \text{largest } p \)

Proof: by induction on \( |P| \).
Base Case: \(|P| = 0\)  \(\checkmark\)

Assume \(|P| \geq 0\)

\(m\) is the size of a largest anti-chain.

\(C = x_1 < x_2 < \cdots < x_m\)

is a longest chain.

Case 1: Every anti-chain in \(P \setminus C\) has \(\leq m-1\) elements. We use induction to find \(C_j, C_2, \ldots, C_{m-1}\) & add \(C\).
Case 2: An anti-chain $A := \{a_1, a_2, \ldots, a_\mu\} \subseteq P \setminus C$

\[
\rho^+ \subseteq \{ x \in P : x \geq \text{some } a_i \} \\
\rho^- \subseteq \{ x \in P : x \leq \text{some } a_i \}
\]

(i) $\rho = \rho^+ \cup \rho^-$

(ii) $\rho^+ \cap \rho^- = A$

(iii) $|\rho^-| < |P|$
Proof of Erdős-Szekeres

$a_1, a_2, \ldots, a_{n^2+1}$ contain a monotone sequence of length $n+1$.

$P = \{ (i, a_i) : 1 \leq i \leq n^2+1 \}$

$(i, a_i) < (j, a_j)$ if $a_i \leq a_j$,

Suppose $P$ has no chain of length $n+1$.

$\Rightarrow$ we need $\geq n+1$ chains to cover $P$. 
So there is an anti-chain $\mathcal{A}$ of size $n+1$:

$$(i_1, a_{i_1}) \quad (i_2, a_{i_2}) \quad \ldots \quad (i_{n+1}, a_{i_{n+1}})$$

ordered so that $i_1 < i_2 < \ldots < i_{n+1}$

$$\Rightarrow a_{i_1} > a_{i_{n+1}}$$

else $A$ is not an anti-chain.