Theorem

If $X, Y$ are random variables on a probability space $\Omega$, then

$$E(X + Y) = E(X) + E(Y)$$

$(X + Y)(w) = X(w) + Y(w)$

No assumptions about independence are needed. (Unlike $E(XY) = E(X)E(Y)$ under.)
Binomial Distribution

\[ B(n, p) = \#\text{ of head (} P(\text{H}) = p) \]
when you toss \( n \) coins.

\[ B(n, p) \left( \text{HHHHHHHH} \right) = 4 \]

Write \( B(n, p) = X_1 + X_2 + \ldots + X_n \)

\[ X_i = \begin{cases} 1 & \text{in coin in H} \\ 0 & \text{otherwise} \end{cases} \]

\[ \mathbb{E} \left[ B(n, p) \right] = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \ldots + \mathbb{E}(X_n) \]
\[ = n \mathbb{E}(X_1) = n (1 \times p, 0 \times (1 - p)) = np \]
Same probability space $\{H, T\}^n$

Suppose $Z = \# \emptyset$ hence we get HH

$$Z(HHHHTHTHHHT) = 3$$

$$Z_i = \begin{cases} 1 & \text{if } i, i+1 \text{ coin } H \\ 0 & \text{otherwise} \end{cases}$$

$$Z = Z_1 + Z_2 + \cdots + Z_{n-1}$$

$$E(Z) = E(Z_1) + \cdots + E(Z_{n-1})$$

$$= \rho^2 + \cdots + \rho^2$$

$$= (n-1) \rho^2.$$
\( M \) indistinguishable balls
\( N \) colors
\( Z = \# \text{ colors used} \)
\( Z_i = \begin{cases} 1 & \text{color } i \text{ is used} \\ 0 & \text{otherwise} \end{cases} \)
\[ E(Z) = n \ E(Z_i) \]
\[ = n \ \Pr(Z_i \neq 0) \]
\[ = n \ (1 - \Pr(Z_i = 0)) \]
\[ = n \left( 1 - \frac{\binom{n+m-2}{m}}{\binom{n+m-1}{m}} \right) \]
\[
= n \left( 1 - \frac{\binom{n+m-2}{m}}{\binom{n+m-1}{m}} \right)
\]

\[
= n \left( 1 - \frac{(n+m-2)!}{m! \cdot (n-2)!} \cdot \frac{m! \cdot (n-1)!}{(n+m-1)!} \right)
\]

\[
= n \left( 1 - \frac{n-1}{n+m-1} \right)
\]

\[
= \frac{m \cdot n}{n + m - 1}
\]
4 BALLS
3 COLORS

4 outcomes

RRRB
RRBR
RBBR
BRBR

1 outcome

DISTINGUISHABLE

INDISTINGUISHABLE
HAT PUZZLE

1) I can see the hat of 3, 4, 5, 6.

2) The color of the hats are random.

Each person says:
(i) My hat is RED
(ii) My hat is BLUE
(iii) I don’t know.

Big prize if one person correctly guesses own hat color & no one is wrong.

Big penalty if someone is wrong.

Is there a strategy that wins with "high" prob.
Claim: Can win with probability \( \geq 1 - \frac{268n}{n} \).

Represent hat colors by \( \{0, 1\} \rightarrow Q_n \)

\[
Q_n = W \cup L
\]

I.e.,

\[
W \cup L = \{0, 1\}^n
\]

L has property that if \( x = x_1 x_2 \ldots x_n \in L \) then can change one bit \( x_i \), i say, so that \( x, x_2 \ldots x_{i-1} x' x_{i+1} \ldots x_n \in L \).
Before the hats are placed, people agree on an L (remember).

Person i: $x_i = ?$

I will assume $x \in \mathbb{W}$.

If, say, $x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n \in \mathbb{W}$

$x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n \notin \mathbb{L}$

I will say $x_i = 0$. 
How to choose $L$.

$p = \frac{\log n}{n}$

Choose $L_1$ randomly by placing $x \in \Theta_n$ into $L_1$ with probability $p$.

$L_2 = \{x \in x \text{ not covered by } L_1\}$

$L = L_1 \cup L_2$

$E(1_{L_1}) = 2^n p + 2^n (1-p)^{n+1}$

$E(1_{L_1}) \quad E(1_{L_2})$