Department of Mathematics  
Carnegie Mellon University  

21-301 Combinatorics, Fall 2006: Test 4

Name: ________________________________

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Q1: (33pts) A is an $n \times n$ matrix with entries 0 or 1. Let $k$ be a positive integer. Show that if $n \geq R(2k, 2k)$ then there exists a set of rows $I$ and columns $J$ such that (i) $|I| = |J| = k$ and (ii) $i, i' \in I$ and $j, j' \in J$ implies $A(i, j) = A(i', j')$.

Solution This is HW9, Q2. Given $A$ we construct a coloring $\tau$ of the edges of $K_n$ as follows. If $i < j$ then we give the edge $(i, j)$ of $K_n$ the color Red if $A_{i,j} = 0$ and Blue if $A_{i,j} = 1$.

Since $n \geq R(2k, 2k)$ we see that $K_n$ contains a mono-colored copy of $K_{2k}$. If the set of vertices of this copy is $S$, divide $S$ into two parts $S_1, S_2$ of size $k$ where $\max S_1 < \min S_2$. It follows that the sub-matrix given by $I = S_1, J = S_2$ satisfies our requirements.
Q2: (33pts) $A_1, A_2, \ldots, A_{mn+1}$ are non-empty subsets of $[n]$. Show that there exists $I \subseteq [mn+1]$ such that (i) $|I| = m+1$ and (ii) if $i, j \in I$ then $A_i \nsubseteq A_j$ and $A_j \nsubseteq A_i$.

Solution Consider the poset on $\{A_1, A_2, \ldots, A_{mn+1}\}$ with $\leq$ equal to $\subseteq$. The maximum length of a chain $X_1 \subset X_2 \subset \cdots \subset X_k$ in this poset is at most $n$, since $|X_k| \geq k$. Applying Dilworth’s theorem, we see that there is an anti-chain $\{A_i : i \in I\}$ of size $\lceil (mn+1)/n \rceil = m+1$. 


Q3: (34pts) Consider the following game: There is a pile of \( n \) chips. A move consists of removing \( 3^k \) chips for some \( k \geq 0 \).

(a) Compute the Sparague-Grundy numbers \( g(n) \) for \( n = 0, 1, 2, \ldots, 10 \).

(b) Make a guess at the general form of \( g(n) \).

(c) Give an inductive proof of your conjecture in (b).

Solution

\[
\begin{array}{cccccccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
g(n) & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

(b) \( g(n) = n \mod 2 \).

(c) This is true for \( 0 \leq n \leq 10 \). Because \( 3^i \) is odd for \( i \geq 0 \) we see that for \( k > 5 \) we have

\[
g(2k - 1) = \text{mex}\{g(2k - 1 - 3^i) : i \geq 0, 3^i \leq 2k - 1\} = \text{mex}\{0, 0, \ldots, 0\} = 1.
\]

\[
g(2k) = \text{mex}\{g(2k - 3^i) : i \geq 0, 3^i \leq 2k\} = \text{mex}\{1, 1, \ldots, 1\} = 0.
\]