Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2005: Test 2

Name: ____________________________

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**Q1: (33pts)** A box has four drawers; one contains three gold coins, one contains two gold coins and a silver coin, one contains a gold coin and two silver coins and one contains three silver coins. Assume that one drawer is selected randomly and that a randomly selected coin from that drawer turns out to be gold. What is the probability that the chosen drawer is the one with three gold coins?

**Solution:** Let the four drawers be $A, B, C, D$. Let $G, S$ stand for the chosen coin being Gold/Silver respectively. Then what we want is

$$Pr(A \mid G) = \frac{Pr(A \land G)}{Pr(G)}.$$

Now

$$Pr(A \land G) = Pr(A) = \frac{1}{4}.$$

$$Pr(G) = Pr(G \mid A) Pr(A) + Pr(G \mid B) Pr(B) + Pr(G \mid C) Pr(C) + Pr(G \mid D) Pr(D)$$

$$= 1 \times \frac{1}{4} + \frac{2}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{1}{4} + 0 \times \frac{1}{4}$$

$$= \frac{1}{2}.$$

So

$$Pr(A \mid G) = \frac{1/4}{1/2} = \frac{1}{2}.$$
Q2: (33pts) Let $A_1, A_2, \ldots, A_n$ be subsets of $A$ with $|A_i| = k$ for $1 \leq i \leq n$. Show that if $n(2^{1-k} + k2^{1-k}) < 1$ then it is possible to partition the set $A$ into two sets $R, B$ (i.e. color $A$ red and blue) so that

$$|A_i \cap R| \geq 2 \quad \text{and} \quad |A_i \cap B| \geq 2 \quad \text{for} \quad i = 1, 2, \ldots, n.$$ 

Solution Let $\mathcal{E}_{i,X}$ be the event that color $X$ is not used twice on $A_i$ and let $\mathcal{E}_i = \mathcal{E}_{i,R} \cup \mathcal{E}_{i,B}$. Then

$$\Pr(\mathcal{E}_i) \leq \Pr(\mathcal{E}_{i,R}) + \Pr(\mathcal{E}_{i,B}) = 2^{1-k} + \binom{k}{1} 2^{-k} = 2^{1-k} + k2^{1-k}.$$

Thus,

$$\Pr\left(\bigcup_{i=1}^{n} \Pr(\mathcal{E}_i)\right) \leq n(2^{1-k} + k2^{1-k}) < 1$$

and so there is a coloring for which none of the $\mathcal{E}_i$ occur.
Q3: (34pts)
A particle sits at the left hand end of a line $0 - 1 - 2 - \cdots - L$. When at 0 it moves to 1. When at $i \in [1, L - 1]$ it makes a move to $i - 1$ with probability 1/3 and a move to $i + 1$ with probability 2/3. When at $L$ it stops.
Let $E_k$ denote the expected number of visits to 0 if we started the walk at $k$.

1. Find a set of equations satisfied by the $E_k$.

2. Given that $E_k = \frac{A}{3^k} + B$ is a solution to your equations for some $A, B$, determine $A, B$ and hence find $E_0$.

Solution: The equations are

\[
E_L = 0 \\
E_0 = 1 + E_1 \\
E_k = \frac{1}{3}E_{k-1} + \frac{2}{3}E_{k+1}
\]

for $0 < k < L$.

$E_L = 0$ implies then that $\frac{A}{3^L} + B = 0$ and so $B = -\frac{A}{3^L}$.

$E_0 = 1 + E_1$ implies then that $A + B = 1 + \frac{1}{2}A + B$ which implies that $A = 2$.
Thus $E_0 = 2 - 2^{1-L}$. 