Roth's Theorem

Fix $0 < \delta < 1$. If n is large enough and $A \subseteq [n]$, $|A| = \delta n$ then

A contains x, y, z s.t. x, y, z form a 3-term arithmetic progression

$$y = \frac{1}{2}(x+z).$$

$\delta = \frac{c}{\log\log n}$ works here.

Combinational Proof

Replace $[1, n]$ by $\mathbb{Z}_n$:

Divide $[1, n]$ into $\underset{I_1}{[1, n/4]}, \underset{I_2}{(\frac{n}{4}, \frac{n}{2}]}, \underset{I_3}{(\frac{n}{2}, \frac{3n}{4})}, \underset{I_4}{(\frac{3n}{4}, n]}$.

$A_j = A \cap I_j$

Choose j such that $|A_j| \geq |A|/4$ and then

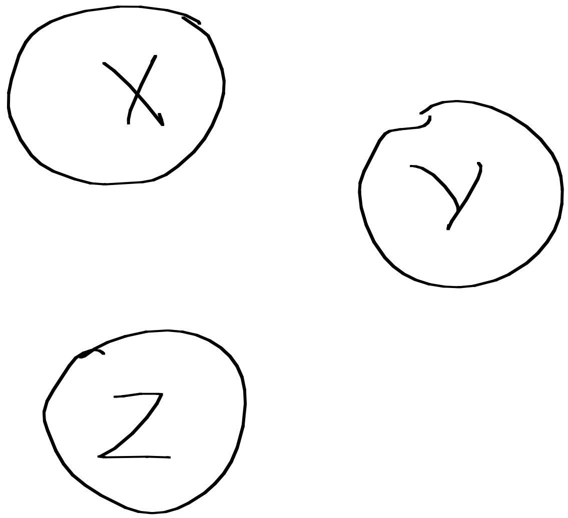
$$A \leftarrow A_j - \{(j-1)\, n/4\}.$$

Now any 3 term AP of A in $\mathbb{Z}_n$ is also a 3-term AP in $[1, n]$.

Triangle Removal Lemma

If $G = (V, E)$, $|V| = n$ and G contains $\geq c_1 n^2$ edge disjoint Δ's then it contains $\geq c_2 n^3$ Δ's $\quad c_2 = c_2(c_1)$.

Roth



Edges: $(x,y): \exists a \in A: x+a=y$

$(x,z): \exists b \in A: x+b+b=z$

$(y,z): \exists c \in A: y+c=z$

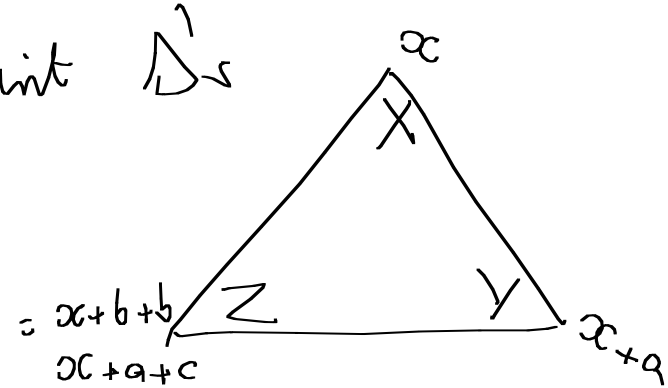
$\forall x \in V,\quad x,\quad x+a,\quad x+a+a$ in Δ

$\quad\quad X \quad\quad Y \quad\quad Z$

$\Longrightarrow\ \geq \delta n^2$ edge disjoint Δ's

$\Longrightarrow\ \exists\ \delta' n^3\ \Delta$'s

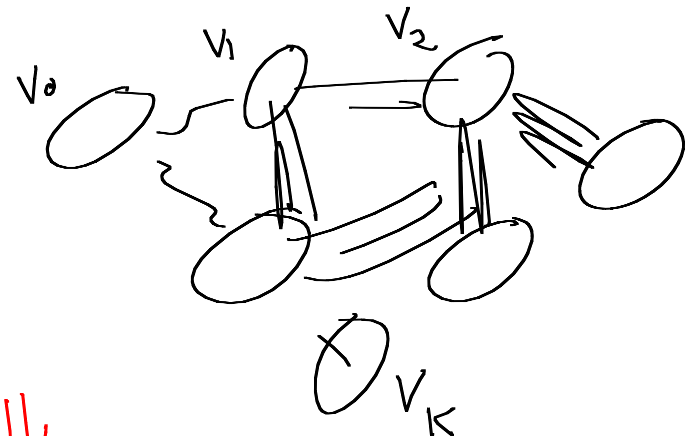
$\exists\ \Delta$:



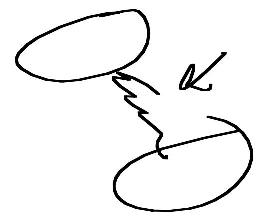
Triangle Removal Lemma

Assume c is small.

G: apply Szemerédi Lemma: $\frac{c}{4}$-regular partition:



Remove following edges:

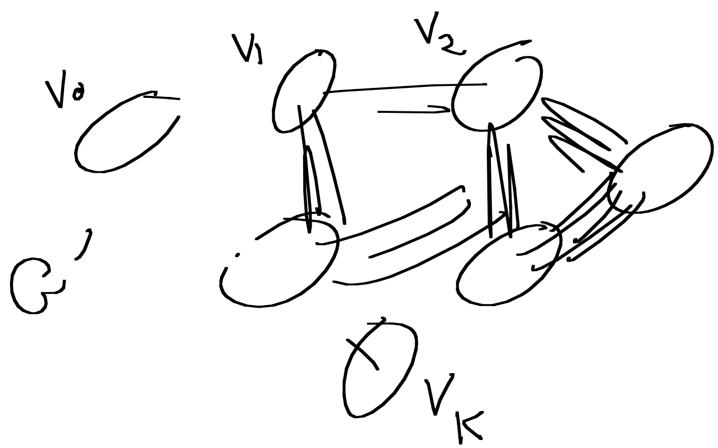


(i) irregular

(ii) $\leq \frac{c}{2}$ dense

edges removed $\leq cn^2$

$\Downarrow$



G' is either (i) Δ-free

or

(ii) has $\frac{c^3 n^3}{256 k^3}$ Δ's

So if $G \geq cn^2$ edge disjoint Δ's

then $G \geq \dfrac{c^3 n^3}{256 K^3}$ Δ's.

Fourier Analysis

Group G.

$$e : G \times G \longrightarrow S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$$

$G = \mathbb{Z}_N$: $\quad e(x, \xi) = e^{2\pi x \xi i / N}$

Properties:

$$e(x + x', \xi) = e(x, \xi)\, e(x', \xi)$$

$$e(x, \xi + \xi') = e(x, \xi)\, e(x, \xi')$$

Properties: (a) $e(0,\xi) = e(x,0) = 1$

$$e(x,-\xi) = e(-x,\xi) = \overline{e(x,\xi)}$$

(b) $\displaystyle \frac{1}{|G|} \sum_{x\in G} e(x,\xi)\, \overline{e(x,\xi')} = 1_{\xi=\xi'}$

Orthonormality

(c) $\displaystyle \hat{f}(\xi) = \frac{1}{|G|} \sum_{x\in G} f(x)\, \overline{e(x,\xi)}$

$f : G \to \mathbb{C}$

$$f(x) = \sum_{\xi \in G} \hat{f}(\xi)\, e(x, \xi)$$ Inversion

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)\, \overline{g(x)}$$

Parseval

$$= \sum_{\xi \in G} \hat{f}(\xi)\, \overline{\hat{g}(\xi)}$$

$$\frac{1}{|G|} \sum_{x \in G} |f(x)|^2 = \sum_{\xi \in G} |\hat{f}(\xi)|^2$$ Plancherel

$$f * g(x) = \frac{1}{|G|} \sum_{y \in G} f(y)\, g(x-y)$$ Convolution

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\, \hat{g}(\xi)$$

$$\tilde{f}(x) = \overline{f(-x)}$$ ← Definition

$$\hat{\tilde{f}}(\xi) = \tilde{\hat{f}}(\xi),$$

Roth's Theorem by Fourier Analysis

Fix $0 < \delta < 1$. $A \subseteq [n]$, $|A| = \delta n$

$\Rightarrow \exists$ 3-term arithmetic progression in A.

Easy Case

$\delta \geq .9$. A must contain $x, x+1, x+2$

else $|A| \leq \frac{2}{3} n$.

"Proof by induction on δ"

Assume $|A|$ is odd and that B larger of even elements / odd elements of A.

χ_A, χ_B indicator functions of A, B.

Reduce to $A \subseteq \mathbb{Z}_n$

$$\overset{B}{x} + \overset{B}{y} = 2\overset{A}{z} \mod n \Rightarrow x + y = 2z + kn$$

$k \in \{0, \pm 1\}$

$$\underbrace{x + y}_{\text{even}} = \underbrace{2z}_{\text{even}} + \underbrace{n}_{\text{odd}}$$ parity problem!

$$\Delta = n^2 \sum_{g \in G} \hat{\chi}_B(g)^2 \, \hat{\chi}_A(-2g)$$

$$= \#(x + y = 2z \mod n, \ x, y \in B, \ z \in A)$$

$$\Delta = \frac{1}{n} \sum_{g \in G} \sum_{\substack{b_1 \in B \\ b_2 \in B \\ a \in A}} \underbrace{\overline{e(b_1, g)} \, \overline{e(b_2, g)} \, \overline{e(-2a, g)}}_{\exp\{2\pi i (2a - b_1 - b_2) g / n\}}$$

Now note

$$\sum_{g \in G} e^{-2\pi i x g / n} = \begin{cases} n & x = 0 \\ 0 & x \neq 0 \end{cases}$$

There $|B|$ trivial AP's where $x=y=z$

$$\Delta - |B| = n^2 \sum_{\substack{g\in G \\ g\neq 0}} \hat{1}_B(g)^2 \, \hat{1}_A(-2g) + \frac{|A|\cdot|B|^2}{n} - |B|$$

$$\left[\hat{1}_A(0) = \frac{|A|}{n}, \quad \hat{1}_B(0) = \frac{|B|}{n} \right]$$

Case 1: $|\hat{\chi}_A(g)| \leq \frac{\delta^2}{4}, \ \forall g \in G, g \neq 0$

$$n^2 \left| \sum_{\substack{g \in G \\ g \neq 0}} \hat{\chi}_B(g)^2 \hat{\chi}_A(-2g) \right| \leq \frac{\delta^2 n^2}{4} \sum_{g \in G} |\hat{\chi}_B(g)|^2$$

$$= \frac{\delta^2 n}{4} \sum_{x \in G} |\chi_B(x)|^2$$

$$= \frac{\delta^2 n |B|}{4} = \frac{|A|^2 |B|}{4n} \leq \frac{|A||B|^2}{2n}.$$

$$\Delta - |B| \geq \frac{1}{2n} |A| \cdot |B|^2 - |B| > 0.$$

DONE

Case 2: $\exists\, g^* \neq 0$: $|\widehat{\chi_A}(g^*)| \geq \delta^2/4$

$$\left|\frac{1}{n}\sum_{x\in G}(\chi_A(x)-\delta)\overline{e(x,g^*)}\right| \geq \delta^2/4$$

Fix $0 < Q < n$: $Q = \sqrt{n}$

Dirichlet theorem [PHP]

$\exists\, b/q,\ q \leq Q,\ (b,q)=1$: $\left|\frac{g^*}{n} - \frac{b}{q}\right| \leq \frac{1}{qQ}$

Divide $[0, n-1]$ into progressions mod q, each of length $\approx n/q$

Divide each progression into $M = \Theta(\sqrt{n})$ contiguous pieces

Fix an interval I and $x \in I$

$$\overline{e(x, g^*)} = \exp\{-2\pi i x g^*/n\}$$
$$= \exp\left\{-2\pi i x\left(\frac{b}{q} + \frac{\epsilon}{qQ}\right)\right\} \quad |\epsilon| \leq 1$$

$x' \in I \Rightarrow x' = x + rq$, integer r, $|r| \leq \frac{n}{qM}$

$$\frac{\overline{e(x', g^*)}}{\overline{e(x, g^*)}} = \exp\left\{2\pi i\left(br + \frac{\epsilon r}{Q}\right)\right\}$$
$$= \exp\{2\pi i \cdot \epsilon r/Q\}$$
$$= 1 + O\left(\frac{n}{qQM}\right)$$

$$\frac{n\delta^2}{4} \le \sum_I \left| \sum_{x \in I} (\chi_A(x) - \delta) \overline{e(x, q^*)} \right|$$

$$\le \sum_I \left\{ \left| \sum_{x \in I} (\chi_A(x) - \delta) \right| + O\left(\frac{n|I|}{qQM} \right) \right\}$$

$$= \sum_I \left| \sum_{x \in I} (\chi_A(x) - \delta) \right| + O\left(\frac{n^2}{qQM} \right)$$

|| kill

$Q = \sqrt{n}$, $M = C\sqrt{n}/(q\delta^2)$ $\quad$ C is large

$$\frac{n\delta^2}{8} \le \sum_I \left| \sum_{x \in I} (\chi_A(x) - \delta) \right|$$

$$\frac{n\delta^2}{8} \leqslant \sum_I \left| \sum_{x \in I} (\chi_A(x) - \delta) \right|$$

$$\sum_I \sum_{x \in I} (\chi_A(x) - \delta) = 0$$

and so

$$\exists I : \sum_{x \in I} (\chi_A(x) - \delta) \geqslant \frac{\delta^2 n}{16 q M} \qquad \# I = \frac{n}{qM}$$

$$\frac{|A \cap I|}{|I|} \geqslant \delta + \frac{\delta^2}{16}$$

Translate and dilate to $\left[1, \frac{n}{qM}\right]$

$$\delta \longrightarrow \delta\left(1+\frac{\delta}{16}\right)$$

$$n \longrightarrow \frac{n}{qM} = \delta^2\sqrt{n}/c$$

Iterate $L = D/\delta$ times

Density: $\nearrow \geq \delta\left(1+\frac{\delta}{16}\right)^{D/\delta}$ $\quad \delta \gg \frac{1}{\log\log n}$

Size: $\delta^2 n^{\frac{1}{2^L}}/C \gg 100$

Behrend's Theorem

$\exists A \subseteq [1, N]$, $|A| \geq n e^{-c\sqrt{\log n}}$, A has no 3-term progressions.

Proof

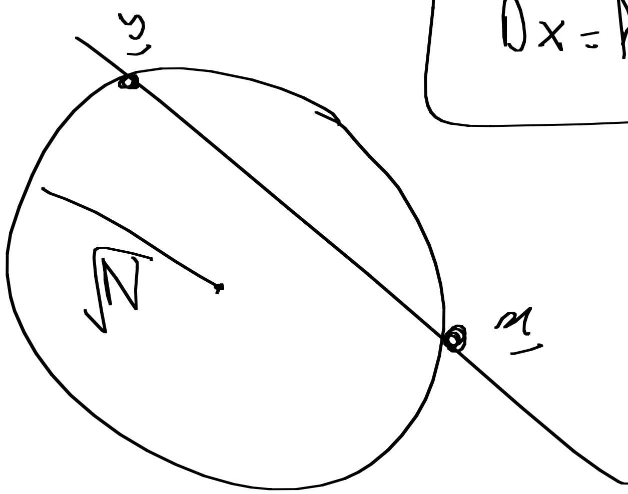
Consider $(x_1, x_2, \dots x_k) \in [0, d]^k$ integer vectors

$$\sum_{i=0}^{k} x_i^2 \in [0, kd^2]$$

$$\exists N \leq kd^2 : \sum_{i=1}^{k} x_i^2 = N \text{ at least } \frac{(d+1)^k}{kd^2} \text{ times}$$

$$A = \left\{ \sum_{i=1}^{k} x_i (2d+1)^{i-1} : \sum_{i=1}^{k} x_i^2 = N \right\}$$

$$\boxed{Dx = N \text{ meets } \bigcirc}$$



AP $\Rightarrow$

Check each expression is different.