4-term arithmetic progressions

Gowers - Szemerédi.

$A \subseteq \{0, 1, \ldots, N-1\} = \mathbb{Z}_N$

$|A| \geq \delta n$ [δ constant]

$\Rightarrow$ A contains a 4-term arithmetic progression.

We can also have $A \subseteq [1, 2, \ldots, N]$

Argument as for Roth's Theorem

Uniformity

$f : \mathbb{Z}_N \to D = \{ z \in \mathbb{C} : |z| \leq 1 \}$

$\alpha > 0$

f is α-uniform if

$$\sum_{\xi} | \hat{f}(\xi) |^4 \leq \alpha$$

$$\hat{f}(\xi) = \frac{1}{N} \sum_{x=0}^{N-1} f(x)\, \omega^{x\xi} \qquad \omega = e^{-2\pi i/N}$$

For $A \subseteq \mathbb{Z}_N$, $A(x)$ is its characteristic function. $|A| = \delta N$.

A is α-uniform if $f(x) = A(x) - \delta$ is α-uniform.

Observation

A is α-uniform if $\sum_{\xi} |\hat{A}(\xi)|^4 \leq \delta^4 + \alpha$

$$\left[\; \hat{A}(\xi) = \begin{cases} \hat{f}(\xi), & \xi \neq 0 \\ \underbrace{\hat{f}(0)}_{=0} + \delta & \xi = 0 \end{cases} \right.$$

$$\left[\; \begin{array}{l} \text{If } g(x) = \delta \\ \quad \forall x \text{ then} \\ \hat{g}(\xi) = 1_{\xi = 0} \end{array} \right.$$

Now define

$$\Delta(f;k)(x) = f(x)\,\overline{f(x-k)}$$

f is quadratically α-uniform if

$$\sum_k \sum_\xi \left| \widehat{\Delta}(f;k)(\xi) \right|^4 \leqslant \alpha N$$

Also

$$\Delta(f;k_1,k_2,\dots,k_r) = \Delta\big(\Delta(f;k_1,k_2,\dots,k_{r-1});k_r\big)$$

Ordering of k_i does not matter

$$\Delta(f;k_1,k_2)(x) = f(x)\,\overline{f(x-k_1)}\,\overline{f(x-k_2)}\,f(x-k_1-k_2).$$

Proposition

For any g,

$$\sum_{\xi} |\hat{g}(\xi)|^4 = \sum_{\xi} \left| \widehat{g*g}(\xi) \right|^2$$

$\widehat{g*g} = \hat{g}^2$

$$= \frac{1}{N} \sum_{y} |g*g(x)|^2$$

$$= \frac{1}{N} \sum_{y} \left| \frac{1}{N} \sum_{x} g(x)\, g(y-x) \right|^2$$

$$= \frac{1}{N^3} \sum_{y,x,x'} g(x)\, g(y-x)\, \bar{g}(x')\, \bar{g}(y-x')$$

$l = x - x'$
$m = x + x' - y$

$$= \frac{1}{N^3} \sum_{x,l,m} g(x)\, \overline{g(x-l)}\, \overline{g(x-m)}\, g(x-l-m)$$

So if $g : \mathbb{Z}_N \to D$, $\sum_{\xi} |\hat{g}(\xi)|^4 \leq 1$.

Proposition

$$\sum_{k} \sum_{\xi} \left| \widehat{\Delta}(f;k)(\xi) \right|^4$$

$$= \frac{1}{N^3} \sum_{k,x,l,m} \Delta(f;k)(x)\, \Delta(f;k)(x-l-m)\, \overline{\Delta}(f;k)(x-l)\, \overline{\Delta}(f;k)(x-m)$$

$$= \frac{1}{N^3} \sum_{x,k,l,m} \Delta(f;k,l,m)(x)$$

$$= \frac{1}{N^3} \sum_{k,l} \sum_{m} \sum_{x} \Delta(f;k,l)(x)\, \overline{\Delta}(f;k,l)(x-m)$$

$$= \frac{1}{N^3} \sum_{k,l} \left| \sum_{x} \Delta(f;k,l)(x) \right|^2.$$

Proposition

$f : \mathbb{Z}_N \to D$.

f is α-uniform iff for any $g : \mathbb{Z}_N \to \mathbb{C}$

$$\sum_y \Big| \sum_x f(x)\, \overline{g(y-x)} \Big|^2 \leq \sqrt{\alpha}\, N^2 \underbrace{\|g\|_2^2}_{\sum_x |g(x)|^2} \quad (1)$$

Also

f is α-uniform if

$$\max_\xi |\hat{f}(\xi)| \leq \alpha^{1/2}.$$

Proof

$$\sum_y \left| \sum_x f(x)\, \overline{g(y-x)} \right|^2$$

$$= N^2 \sum_y \left| f * g\,(y) \right|^2$$

$$= N^3 \sum_\xi \left| \widehat{f * g}\,(\xi) \right|^2$$

$$= N^3 \sum_\xi |\hat{f}(\xi)|^2 \, |\hat{g}(\xi)|^2$$

$$\leq N^3 \left(\sum_\xi |\hat{f}(\xi)|^4 \right)^{1/2} \left(\sum_\xi |\hat{g}(\xi)|^4 \right)^{1/2} \quad \text{CS}$$

$$\leq N^3 \left(\sum_\xi |\hat{f}(\xi)|^4 \right)^{1/2} \sum_\xi |\hat{g}(\xi)|^2$$

$$= N^2 \left(\sum_\xi |\hat{f}(\xi)|^4 \right)^{1/2} \sum_x |g(x)|^2 .$$

So $> ①$ is immediate if f is α-uniform

On the other hand, putting $g = \overleftarrow{f}$ into $> ①$

we get:

$$\sqrt{\alpha}\, N \geq \sqrt{\alpha} \sum_y |f(y)|^2 \geq \sum_y |f * f(y)|^2$$
$$= N \sum_\xi |\widehat{f * f}(\xi)|^2 = N \sum_\xi |\hat{f}(\xi)|^4.$$

$$\sum_y \Big| \sum_x f(x) f(y-x) \Big|^2 \leq \sqrt{\alpha}\, \|f\|_2^2 \quad ①$$

Finally suppose $\max_{\xi} |\hat{f}(\xi)| \leq \alpha^{1/2}$

then

$$\sum_{\xi} |\hat{f}(\xi)|^4 \leq \alpha \sum_{\xi} |\hat{f}(\xi)|^2$$

$$= \frac{\alpha}{N} \sum_{n} |f(n)|^2$$

$$\leq \alpha.$$

Proposition

If f is quadratically α-uniform then f is $\alpha^{1/2}$-uniform.

Proof

$$\Big(\sum_{\xi} |\hat{f}(\xi)|^4\Big)^2 \overset{PS}{=} \frac{1}{N^6}\Big(\sum_{x,l,m} f(x)\overline{f(x-l)}\,\overline{f(x-m)}\,f(x-l-m)\Big)^2$$

$$= \frac{1}{N^6}\Big(\sum_{x,l,m} \Delta(f;l,m)(x)\Big)^2$$

$$\leq \frac{1}{N^4}\sum_{l,m}\Big|\sum_{x}\Delta(f;l,m)(x)\Big|^2 \quad \text{CS}$$

$$= \frac{1}{N}\sum_{l,\xi}|\widehat{\Delta}(f;l)(\xi)|^4 \quad \text{Prop. 6}$$

$$\leq \alpha.$$

Proposition

Let $f_1, f_2, f_3 : \mathbb{Z}_N \to D$. Suppose f_3 is α uniform.

Then

$$\left| \sum_{a,d} f_1(a)\, f_2(a+d)\, f_3(a+2d) \right| \le \alpha^{1/4} N^2$$

Proof

$$= \left| \sum_{a+c=2b} f_1(a)\, f_2(b)\, f_3(c) \right|$$

$$= \left| N^2 \underbrace{\sum_{\xi} \hat{f}_1(\xi)\, \hat{f}_2(-2\xi)\, \hat{f}_3(\xi)}_{\frac{1}{N^3} \sum_{\xi} \sum_x f_1(x)\, \omega^{x\xi} \sum_y f_2(y)\, \omega^{-2y\xi} \sum_z f_3(z)\, \omega^{z\xi}} \right|$$

$$\sum_{\xi} \sum_{x-2y+z=0} f_1(x)\, f_2(y)\, f_3(z)$$

N or 0

$$= \left| N^2 \sum_{\xi} \hat{f}_1(\xi)\, \hat{f}_2(-2\xi)\, \hat{f}_3(\xi) \right|$$

$$\le \underbrace{N^2 \max_{\xi} \hat{f}_3(\xi)}_{\le \alpha^{1/4}} \underbrace{\Big(\sum_{\xi} |\hat{f}_1(\xi)|^2 \Big)^{1/2}}_{\le 1} \underbrace{\Big(\sum_{\xi} |\hat{f}_2(\xi)|^2 \Big)^{1/2}}_{\le 1} \quad \text{CS}$$

Proposition 7

$$= \frac{1}{N} \sum_{x} |f_j(x)|^2$$

$\square$

Proposition

Let $f_1, f_2, f_3, f_4 : \mathbb{Z}_N \to D$ and $\alpha > 0$. Suppose f_4 is quadratically α-uniform. Then

$$\left| \sum_{a,d} f_1(a)\, f_2(a+d)\, f_3(a+2d)\, f_4(a+3d) \right| \leq \alpha^{1/8} N^2$$

Proof

$$\left| \quad \right|^2 \leq N \sum_a \left| \sum_d \underbrace{f_1(a)}_{\text{indep. of } d} f_2(a+d)\, f_3(a+2d)\, f_4(a+3d) \right|^2 \qquad \text{CS}$$

$$\leq N \sum_a \left| \sum_d f_2(a+d)\, f_3(a+2d)\, f_4(a+3d) \right|^2$$

$$= N \sum_a \sum_{d,e} f_2(a+d)\, \overline{f_2(a+e)}\, f_3(a+2d)\, \overline{f_3(a+2e)}\, f_4(a+3d)\, \overline{f_4(a+3e)}$$

$k = d - e$

$$= N \sum_a \sum_{d,k} \Delta(f_2; k)(a+d)\, \Delta(f_3; 2k)(a+2d)\, \Delta(f_4; 3k)(a+3d)$$

$$= N \sum_{a} \sum_{d,k} \Delta(f_2;k)(a+d)\,\Delta(f_3;2k)(a+2d)\,\Delta(f_4;3k)(a+3d)$$

$a+d \to a$

$$= N \sum_{a} \sum_{d,k} \Delta(f_2;k)(a)\,\Delta(f_3;2k)(a+d)\,\Delta(f_4;3k)(a+2d)$$

Since f_4 is quadratically α-uniform, there are α_k, $k \in \mathbb{Z}_N$ such that $\Delta(f_4;k)$ is α_k-uniform and $\sum_k \alpha_k \leq \alpha N$.

$$\sum_{k} \underbrace{\sum_{r} |\widehat{\Delta}(f;k)(r)|^4}_{\alpha_k} \leq \alpha N$$

Proposition 12 implies

$$\sum_{d,k} \Delta(f_2;k)(a)\,\Delta(f_3;2k)(a+d)\,\Delta(f_4;3k)(a+2d) \leq \alpha_k^{1/4} N^2$$

$$|\sim|^2 \leq \sum_k \alpha_k^{1/4} N^3 \leq \alpha^{1/4} N^4.$$

$$\sum \alpha_k \leq \alpha N$$

$$\text{Max} \sum_k \alpha_k^{1/4} \quad \text{s.t.} \quad \sum \alpha_k \leq \alpha N$$

$$\text{Put } \alpha_k = \alpha$$

Proposition

Let $A_1, A_2, A_3, A_4 \subseteq \mathbb{Z}_N$ with $|A_i| = \delta_i N$.
Suppose that A_3 is $\alpha^{1/2}$ uniform and A_4 is quadratically α-uniform. Then

$$\left| \sum_{a,d} \underline{A_1(a)\, A_2(a+d)\, A_3(a+2d)\, A_4(a+3d)} - \delta_1\delta_2\delta_3\delta_4 N^2 \right| \leq 12\alpha^{1/8} N^2$$

Proof

Write $A_i(x) = f_i(x) + \delta_i$. Sum splits into 16 parts.

Case 1: f_4 used: Apply Proposition 14. Total $\leq 8\alpha^{1/8} N^2$

Case 2: δ_4, f_3 used: Apply Proposition 12. Total $\leq 4(\alpha^{1/2})^{1/4} N^2$

Case 3: δ_4 & δ_3 used: $\sum_{a,d} A_1(a) f_2(a+d) = \left(\sum_a A_1(a)\right)\left(\sum_b f_2(b)\right) = 0$

Case 4: Only δ_i's used: $\delta_1\delta_2\delta_3\delta_N N^2$

Proposition

$A \subseteq \delta N$, $|A| = \delta N$. A is quadratically α-uniform.

If $\alpha \leq \delta^{32}/2^{88}$ and $N \geq 200/\delta^4$ then either

(i) A contains an AP of length 4, or

(ii) there is a sub-progression on which A has density $\geq \frac{9\delta}{8}$.

Proof

$A_1 = A_2 = A \cap [2N/5, 3N/5]$ and $A_3 = A_4 = A$.

If $|A_1| \leq \frac{\delta}{10} N$ then either (i) $|A \cap [0, 2N/5]| \geq \frac{9\delta}{20} N$

or (ii) $|A \cap [3N/5, N]| \geq \frac{9\delta}{20} N$

density $\geq \frac{9}{20} \times \frac{5}{2} \delta$

Suppose $|A_1| \geq \frac{\delta}{10} N$.

Applying Proposition 17 we see that $A_1 \times A_2 \times A_3 \times A_4$ contains at least $(\delta^4/100 - 12\alpha^{1/8}) N^2$ $\mathbb{Z}_N$ A.P.'s of length 4.

To rule out progressions with $d = 0$ we ensure

$$(\delta^4/100 - 12\alpha^{1/8}) N^2 > \delta N.$$

Any such progression with $d \neq 0$ is also a $\mathbb{Z}$ A.P.

$d < N/5$

$d < \frac{N}{5}$ $\quad a, a+d, a+2d, a+3d$

or $\quad a, a-d, a-2d, a-3d$

Proposition

Suppose $f: \mathbb{Z}_N \to D$ is not quadratically α-uniform.
Then $\exists\, B \subseteq \mathbb{Z}_N : |B| > \alpha N/2$ and a function $\phi : B \to \mathbb{Z}_N$ such that

$$\sum_{k \in B} \left| \hat{\Delta}(f;k)(\phi(k)) \right|^2 \geq \frac{\alpha^2}{4} N$$

Proof

$$\sum_{k} \underbrace{\sum_{\xi} \left| \hat{\Delta}(f;k)(\xi) \right|^4}_{\leq 1 \text{ - see p5}} > \alpha N$$

So $\exists \geq \alpha N/2$ values of k for which

$$\sum_{\xi} \left| \hat{\Delta}(f;k)(\xi) \right|^4 \geq \frac{\alpha}{2} \qquad B = \{k\}$$

$$\sum_{\xi} |\hat{\Delta}(f_j k)(\xi)|^4 \geq \frac{\alpha}{2} \qquad B = \{k\}$$

So by Proposition 7

$$\max_{\xi} |\hat{\Delta}(f_j k)(\xi)| > \sqrt{\alpha/2}$$

ξ $\equiv$ $\phi(k) = \text{argmax}$

So

$$\sum_{k \in B} |\hat{\Delta}(f_j k)(\phi(k))|^2 \geq \frac{\alpha N}{2} \cdot \frac{\alpha}{2}.$$

Proposition

Suppose $f : \mathbb{Z}_N \to D$, $B \subseteq \mathbb{Z}_N$ and ϕ is such that

$$\sum_{k \in B} \left| \hat{\Delta}(f;k)(\phi(k)) \right|^2 \geq \alpha N.$$

Then $\exists\ \alpha^4 N^3$ quadruples $(a,b,c,d) \in B^4$ such that

$$a+b = c+d$$
$$\phi(a)+\phi(b) = \phi(c)+\phi(d)$$

additive quadruples for ϕ

Proof

$$\hat{\Delta}(f;k)(\phi(k)) = \frac{1}{N} \sum_x f(x)\overline{f(x-k)}\, \omega^{x\phi(k)}$$

so

$$\sum_{k \in B} \sum_{x,y} f(x)\overline{f(x-k)}\,\overline{f(y)}\, f(y-k)\, \omega^{\phi(k)(x-y)} \geq \alpha N^3$$

$$\sum_{k\in B}\sum_{x,y} f(x)\overline{f(x-k)}\,\overline{f(y)}\,f(y-k)\,\omega^{\phi(k)(x-y)} \geq \alpha N^3$$

$$\Rightarrow \sum_{k\in B}\sum_{x,z} f(x)\overline{f(x-k)}\,\overline{f(x-z)}\,f(x-z-k)\,\omega^{\phi(k)z} \geq \alpha N^3$$

$z = x - y$

drop terms

$$\Rightarrow \sum_{x,z}\left|\sum_{k\in B}\overline{f(x-k)}\,f(x-z-k)\,\omega^{\phi(k)z}\right| \geq \alpha N^3$$

CS

$$\Rightarrow \sum_{z}\left[\sum_{x}\left|\sum_{k\in B}\overline{f(x-k)}\,f(x-z-k)\,\omega^{\phi(k)z}\right|^2\right] \geq \alpha^2 N^4$$

Now write

$$\sum_{x}\left|\sum_{k\in B}\cdots\right|^2 = \gamma(z)N^3$$

$$\Rightarrow \sum_z \left\{ \sum_x \left| \sum_{k\in B} \overline{f(x-k)}\, f(x-z-k)\, \omega^{\phi(k)z} \right|^2 \right\} \geq \alpha^2 N^4 \quad \text{CS}$$

Now write

$$\sum_x \left| \sum_{k\in B} \cdots\cdots \right|^2 = \gamma(z) N^3$$

and so

$$\sum_z \gamma(z) \geq \alpha^2 N$$

$$\Rightarrow \sum_z \gamma(z)^2 \geq \alpha^4 N$$

Now put $g(y) = f(y)\,\overline{f(y-z)}$. Apply Proposition 7 to see that $h_z(k) = B(k)\,\omega^{\phi(k)z}$ is not $\gamma(z)^2$-uniform.

$$\sum_x \left| \sum_k h_z(k) \cdot \overline{g(x-k)} \right|^2 \geq \gamma(z) N^3 \geq \gamma(z) N^2 \|g\|_2^2 .$$

$$\sum_{\xi} \Big| \underbrace{\sum_{k\in B} \omega^{\phi(k) z - \xi k}}_{B(k)\,\omega^{\phi(k) z}} \Big|^4 \geq \gamma(z)^2 N^4$$

is not $\gamma(z)^2$-uniform

Thus

$$\sum_{z} \sum_{\xi} \Big| \qquad \Big|^4 \geq \alpha^4 N^5$$

or

$$\sum_{z} \sum_{\xi} \Big| \sum_{a,b,c,d\in B} \omega^{(\phi(a)+\phi(b)-\phi(c)-\phi(d)) z} \, \omega^{-\xi(a+b-c-d)} \Big|$$

$\geq \alpha^4 N^5$.

But

$\frac{1}{N^2} \times LHS = \# a, b, c, d \in B$ that are additive quadruples.

Proposition

Let $B \subseteq \mathbb{Z}_N$ of cardinality βN and $\phi : B \to \mathbb{Z}_N$ have at least αN^3 additive quadruples. Then $\exists\ \gamma, \eta$ and an arithmetic progression P and linear ψ such that

(i) $|P| \geq N^{\gamma}$

and

(ii) $\psi(s) = \phi(s)$ for $\geq \eta |P|$ values $s \in P$.

Proposition

Let B be as defined on P20. Then $\exists\, \gamma, \eta, P$ and $\psi : P \to \mathbb{Z}_N$ such that

$$\sum_{k \in P} \Big| \hat{\Delta}(f;k)\underbrace{(2\lambda k + \mu)}_{\psi(k)} \Big|^2 \geq \eta |P|$$

Proof

Choose those $k \in P$ for which $\phi(k) = \psi(k)$.

For these k we have

$$|\hat{\Delta}(f;k)(\phi(k))| \geq \sqrt{\alpha/2}$$

$\square$

Proposition

With the assumptions of Proposition 27 and $|P| \leq N^{1/2}$ (drop elements if necessary) there exists a partition of $\mathbb{Z}_N$ into translates $P_1, P_2, \ldots, P_M$ of P (or P with one endpoint removed) such that for each i there exists $r_i \in \mathbb{Z}_N$ such that

$$\sum_i \left| \sum_{x \in P_i} f(x)\, \omega^{\lambda x^2 + r_i x} \right| \geq \eta N/2$$

Proof

$$\sum_{k\in P}\Big|\sum_{x} f(x)\,\overline{f(x-k)}\,\omega^{(2\lambda k+\mu)x}\Big|^2 \geq \eta|P|N^2$$

$\Rightarrow$

$$\sum_{k\in P}\sum_{x}\sum_{y} f(x)\,\overline{f(x-k)}\,\overline{f(y)}\,f(y-k)\,\omega^{(2\lambda k+\mu)(x-y)} \geq \eta|P|N^2$$

$\Rightarrow$

$$\sum_{k\in P}\sum_{x}\sum_{u} f(x)\,\overline{f(x-k)}\,\overline{f(x-u)}\,f(x-k-u)\,\omega^{(2\lambda k+\mu)u} \geq \eta|P|N^2$$

$u = x-y$

Every $u\in\mathbb{Z}_N$ can be written in $|P|$ ways as $v+l$, $v\in\mathbb{Z}_N$ and $l\in P$. So

$$\sum_{\substack{k\in P\\ l\in P}}\sum_{x,v} f(x)\,\overline{f(x-k)}\,\overline{f(x-v-l)}\,f(x-v-k-l)\,\omega^{(2\lambda k+\mu)(v+l)} \geq \eta|P|^2N^2.$$

Hence $\exists v \in \mathbb{Z}_N$ such that

$$\sum_{k\in P} \sum_{l\in P} \sum_{x} f(x)\, \overline{f(x-k)}\, \overline{g(x-l)}\, g(x-k-l)\, \omega^{(2\lambda k+\mu)(v+l)} \geq |P|^2 N$$

where $g(x) = f(x-v)$.

Now let

$$h_1(x) = f(x)\, \omega^{\lambda x^2 + \mu x + \mu v} \qquad h_3(x) = g(x)\, \omega^{\lambda x^2 - (2\lambda v - \mu) x}$$

$$h_2(x) = f(x)\, \omega^{\lambda x^2} \qquad h_4(x) = g(x)\, \omega^{\lambda x^2 - 2\lambda v x}$$

Then

$$\sum_{x} \sum_{k,l\in P} h_1(x)\, \overline{h_2(x-k)}\, \overline{h_3(x-l)}\, h_4(x-k-l) \geq \eta |P|^2 N$$

$$\sum_{x} \underbrace{\sum_{k,\ell \in P} h_1(x)\,\overline{h_2(x-k)}\;\overline{h_3(x-\ell)}\;h_4(x-k-\ell)}_{\eta(x)\,|P|^2} \geq \eta\,|P|^3 N$$

Then

$$\frac{1}{N} \sum_{r} \sum_{m \in P+P} \left| \sum_{k,\ell \in P} \overline{h_2(x-k)}\;\overline{h_3(x-\ell)}\;h_4(x-k-\ell)\,\omega^{r(k+\ell-m)} \right|$$

$$\geq \eta(x)\,|P|^2$$

$$\eta(x)\,|P|^2 N = \sum_{r} \sum_{k,\ell \in P} h_1(x)\,\overline{h_2(x-k)}\;\overline{h_3(x-\ell)}\;h_4(x-k-\ell)$$

$$= \sum_{r} \sum_{m} \sum_{k,\ell \in P} h_1(x)\,\overline{h_2(x-k)}\;\overline{h_3(x-\ell)}\;h_4(x-k-\ell)\,\omega^{r(k+\ell-m)}$$

$$= \sum_{r} \sum_{m \in P+P} \sum_{k,\ell \in P} h_1(x)\,\overline{h_2(x-k)}\;\overline{h_3(x-\ell)}\;h_4(x-k-\ell)\,\omega^{r(k+\ell-m)}$$

$$\leq \sum_{r} \sum_{m \in P+P} \left| \sum_{k,\ell \in P} \cancel{h_1(x)}\,\overline{h_2(x-k)}\;\overline{h_3(x-\ell)}\;h_4(x-k-\ell)\,\omega^{r(k+\ell-m)} \right|$$

$$\frac{1}{N}\sum_{r}\sum_{m\in P+P}\left|\sum_{k,l\in P}\overline{h_2(x-k)}\,\overline{h_3(x-l)}\,h_4(x-k-l)\,\omega^{r(k+l-m)}\right| \geq \eta(x)\,|P|^2$$

$$\Longrightarrow$$

$$\sum_{r}\left|\sum_{k\in P}h_2(x-k)\,\omega^{rk}\right| \times \left|\sum_{l\in P}h_3(x-l)\,\omega^{rl}\right| \times$$

$$\left|\sum_{m\in P+P}h_4(x-m)\,\omega^{-rm}\right| \geq \eta(x)\,|P|^2 N.$$

However,

$$\sum_{r}\left|\sum_{l\in P}h_3(x-l)\,\omega^{rl}\right|^2 = N\sum_{l\in P}\left|h_3(x-l)\right|^2 \leq N|P|$$

$$\sum_{r}\sum_{l,l'}h_3(x-l)\,\overline{h_3(x-l')}\,\omega^{r(l-l')} =$$

and similarly for h_4.

Applying Cauchy-Schwartz,

$$\max_r \left| \sum_{k\in P} h_2(x-k)\, \omega^{rk} \right| \cdot \sqrt{2} \cdot N|P| \geq \eta(x)|P|^2 N$$

$|P+P| \sim 2|P|$

Replacing h_3, h_4 terms by $\sqrt{N|P|}$

So $\exists\, r_x$ such that

$$\left| \sum_{k\in P} h_2(x-k)\, \omega^{r_x k} \right| \geq \eta(x)|P|/\sqrt{2}$$

i.e.

$$\left| \sum_{k\in P} f(x-k)\, \omega^{\lambda(x-k)^2 + r_x(x-k)} \right| \geq \eta(x)|P|/\sqrt{2}$$

$$\left|\sum_{k\in P} f(x-k)\,\omega^{\lambda(x-k)^2 + r_x(x-k)}\right| \geq \eta(x)\,|P|/\sqrt{2}$$

Summing over x

$$\sum_x \left|\sum_{k\in P} f(x-k)\,\omega^{\lambda(x-k)^2 + r_x(x-k)}\right| \geq \eta N\,|P|/\sqrt{2}$$

We have summed over all translates of P. So $\exists$ a partition of $\mathbb{Z}_N$ into translates of value at least $\frac{1}{|P|}$ times this. We get $\frac{1}{\sqrt{2}|P|}$ if we account for dropping elements now and again (to avoid overlap).

For $\phi : \mathbb{Z}_N \to \mathbb{Z}_N$, $\operatorname{diam}_\phi(S) = \max\{|\phi(x) - \phi(y)| : x, y \in S\}$

Proposition

Let $m, r, l \in [N]$ and let P be a $\mathbb{Z}_N$ A.P. of length m. $\phi : \mathbb{Z}_N \to \mathbb{Z}_N$ is a linear function. If $l \leq (m/r)^{1/3}$ then P can be partitioned into P_i of lengths $l, l-1$ such that $\operatorname{diam}_\phi(P_i) \leq \frac{N}{r}$ for each i.

Proof

Assume $P = [0, m-1]$.

$$\begin{array}{ccccc} a, & a+b, & a+2b, & \dots, & a+(m-1)b \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ 0 & 1 & 2 & & m-1 \end{array}$$

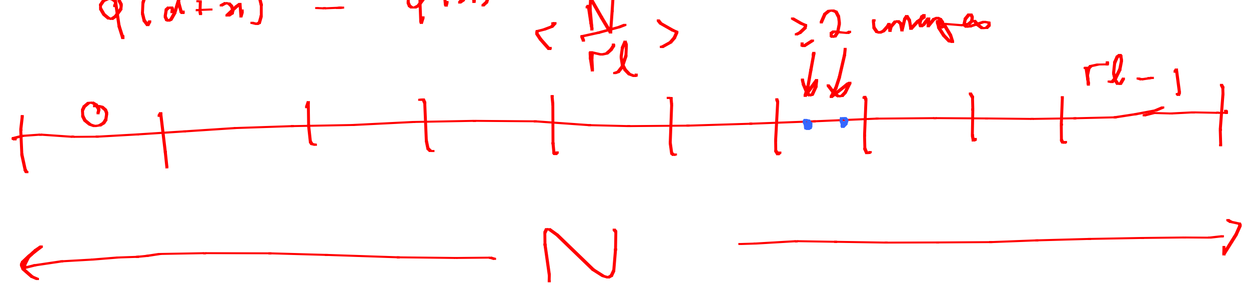
$$\phi(a+kb) = \lambda(a+kb)+\mu$$
$$= (\lambda b)k + (\lambda a+\mu)$$

By Pigeon Hole Principle, $\exists\, d \le rl$ such that

$$|\phi(d) - \phi(0)| \le \frac{N}{rl}$$

— $\phi(0), \phi(1), \dots, \phi(rl)$ gives 2 in a box.

$\phi(d+x) - \phi(x)$



Let $Q_x = \{x, x+d, \dots, x+(l-1)d\}$. Then for $i < j \le l$ we have

$$|\phi(x+jd) - \phi(x+id)| \le (j-i)\,|\phi(d) - \phi(0)|$$
$$\le N/r$$

Observe that $Q_x \cap Q_y = \emptyset$ if $x \not\equiv y \mod d$

Fix $0 \leq x < d$ and consider $\{x, x+d, \dots, x+sd\}$ where $m-d < x+sd \leq m$.

Write
$$s = al - b = (a-b)l + b(l-1)$$
where
$$0 \leq b < l \leq \frac{s}{l} \leq a$$

$$s \geq \frac{m}{d} - 1 \geq \frac{m}{rl} - 1 \geq l^2$$

So P can be divided into a collection $(a-b)$ P_i of length l and b P_i of length $l-1$.

$\square$

Proposition (Weyl)

Given a rational $\frac{a}{N}$ with $(a,N)=1$ and an integer $M \gg 1$, $\exists\, m \leq M$ such that

$$\left\| m^2 \frac{a}{N} \right\| \leq \frac{(\log M)^2}{M^{1/2}}.$$

distance to nearest integer

M does not depend on N here.

Proposition

Let $m \in [N]$ and let $\phi : \mathbb{Z}_N \to \mathbb{Z}_N$ be a quadratic function and let P be an A.P. of length m. Then P can be partitioned into subprogressions P_i, $i \geq 1$ of length l or $l-1$ ($l = m^{\frac{1}{128 \times 18}}$) with $\mathrm{diam}_\phi(P_i) \leq 2m^{-\frac{1}{6 \times 128}} N$.

Proof

Suppose $\phi(x) = ax^2 + bx + c$ and $P = [0, m-1]$.

Choose $d \leq m^{1/2}$ such $|ad^2| \leq m^{-1/5} N \mod N$

(Apply Weyl to $\frac{a}{N}$ to get $d \leq \sqrt{m}$ (M) s.t.

($\| d^2 \frac{a}{N} \| \leq \frac{(\log m)^2}{m^{1/4}}$ .. $\frac{d^2 a}{N} = \underset{0/1}{\delta} + \underset{\leq}{\frac{(\log m)^2}{m^{1/4}}}$))

Let $t = m^{\frac{1}{3 \times 128}}$ and $Q_x = \{x, x+d, \cdots, x+(t-1)d\}$,

$td << m$

Then

$$\phi(x+td) - \phi(x) = (2axd + bd)t + ad^2t^2$$

$$|ad^2t^2| \leq m^{-1/6}N$$

Apply Proposition 35 with $r = m^{\frac{1}{6 \times 128}}$ and $l = m^{\frac{1}{18 \times 128}}$ to see that Q_x can be partitioned into subprogression $R_{x,i}$ of length $l-1, l$

$\left(\frac{m}{l} >> r^3\right)$ $\quad \operatorname{diam} \phi(R_{x,i}) \leq \frac{N}{r} + m^{-1/6}N \leq \frac{2N}{r}$.

We partition P into Q_x's and the Q_x's into $R_{x,i}$'s.

Proposition

Let ϕ be a quadratic function on $\mathbb{Z}_N$ and $r \leq N$. Then $\exists\, m \leq C r^{-\frac{1}{18\times 128}}$ such that $[0, r-1]$ can be partitioned into A.P.'s of lengths differening by ≤ 1 such that if f is any function with

$$\left| \sum_{x=0}^{r-1} f(x)\, \omega^{\phi(x)} \right| \geq \eta r$$

then

$$\sum_{j=1}^{m} \left| \sum_{x \in P_j} f(x) \right| \geq \eta r/2.$$

Proof

By Proposition 39 we can $P_1, P_2, \ldots, P_q$ such that $\operatorname{diam}_\phi (P_i) \leq 2m^{-\frac{1}{6\times 128}} N \leq \frac{\eta N}{4\pi}$ for large enough m. By triangle inequality

$$\sum_{j=1}^{m} \left| \sum_{x \in P_j} f(x)\, \omega^{\phi(x)} \right| \geq \eta r$$

Let $x, y \in P_j$.

$$|\omega^{\phi(x)} - \omega^{\phi(y)}| = |1 - \omega^{\phi(y)-\phi(x)}| \leq \eta/2$$

$$\sum_{j=1}^{m} \Big| \sum_{x\in P_j} f(x) \Big| = \sum_{j=1}^{m} \Big| \sum_{x \in P_j} f(x)\, \omega^{\phi(x_j)} \Big|$$

arbitrary choice $x_j \in P_j$

$$\geq \sum_{j=1}^{m} \Big| \sum_{x\in P_j} f(x)\, \omega^{\phi(x)} \Big| - \sum_{j=1}^{m} \Big| \sum_{x\in P_j} f(x) \left(\omega^{\phi(x)} - \omega^{\phi(x_j)}\right) \Big|$$

$$\geq \eta r - \sum_{j=1}^{m} |P_j| \cdot \eta/2$$

$f : \mathbb{Z}_N \to D$

$$\geq \eta r - \eta r/2$$

$$= \eta r/2 .$$

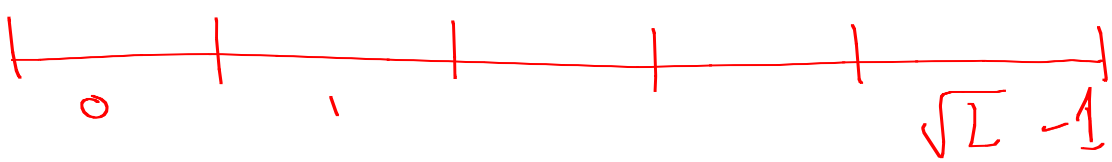
Proposition:

Let P be a $\mathbb{Z}_N$ progression of length $L \gg 1$. Then we can partition P into $\leq 4\sqrt{L}$ genuine $\mathbb{Z}$, A.P.'s.

Proof

$$P = \{a, a+q, \dots, a+Lq\}$$

By PHP, $\exists\, l \leq \sqrt{L}$ with $\left\|\frac{lq}{N}\right\| \leq \frac{1}{\sqrt{L}}$



We split P into progressions with common difference lq.

P_0: $a, a+lq, a+2lq$ length $L/l \geq \sqrt{L}$

P_1: $a+q, a+(l+1)q, \ldots$

$\vdots$

P_{l-1}: $a+(l-1)q, a+(2l-1)q, \ldots$

Now $lq = mN + fN$ where $m \in \mathbb{Z}$ & $0 \leq f \leq \frac{1}{\sqrt{L}}$

So, concentrating on P_0 say if

$$a+(i-1)lq \mod N > a+ilq \mod N$$

then $a+ilq \mod N, a+(i+1)lq \mod N, \ldots, a+(i+(l-1))lq \mod N$ forms $\mathbb{Z}$ A.P.

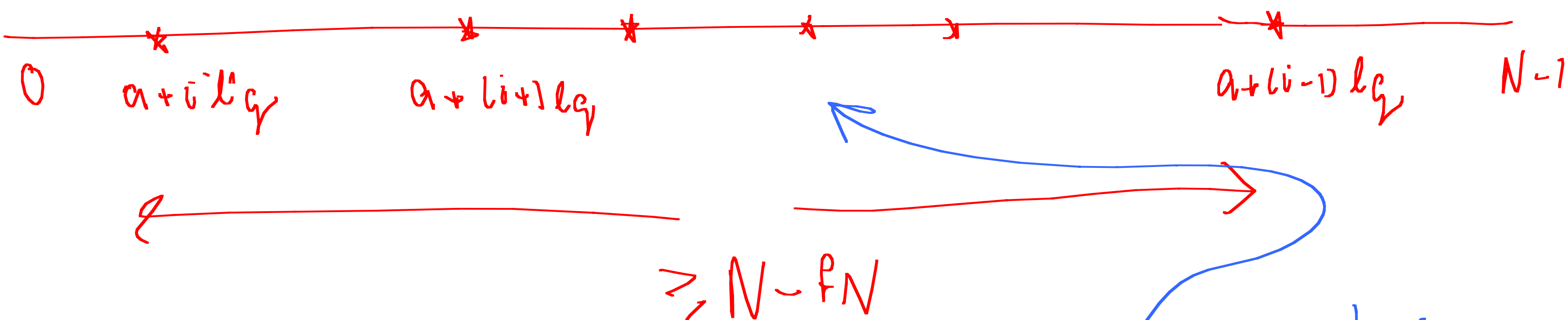
Now $l_q = mN + fN$ where $m \in \mathbb{Z}$ and $0 \le f \le \frac{1}{\sqrt{L}}$

So, concentrating on P_0, say if

$$a + (i-1)l_q \bmod N > a + il_q \bmod N$$

then $a + il_q \bmod N,\ a + (i+1)l_q \bmod N, \ldots, a + (i+(L-1))l_q \bmod N$

form a $\mathbb{Z}$ A.P.



$$\# \text{points} \geq \frac{N-fN}{2l_q}$$

$\frac{\text{Initial}}{\text{Map} \geq \frac{1}{2}N/l_q}$

$\square$

Putting it all together

$\exists$ an absolute constant $c > 0$ such that if $A \subseteq [N]$, $|A| = \delta N$ and $N \gg 1$ and $\delta \geq 1/(\log\log\log N)^c$ then A contains a 4-term A.P.

Proof

If A is quadratically $\alpha = \delta^{32}/2^{88}$ uniform then done by Proposition 17.

Suppose A is not quadratically α-uniform.

By Proposition 20, $\exists B$, $|B| \geq \alpha N/2$ and $\phi : B \to \mathbb{Z}_N$ s.t.:

$$\sum_{k \in B} |\hat{\Delta}(f;k)(\phi(k))| \geq \frac{\alpha^2}{4} N.$$

By Proposition 22 B, ϕ has at least $\left(\frac{\alpha}{2}\right)^3 N^3$ additive quadruples and so by Propositions 26 and 27 there exists A.P. P, $|P| \geq N^{\gamma}$ and

$$\sum_{k \in P} |\hat{\Delta}(f;k)\underbrace{(2\lambda k + \mu)}_{\psi(k)}|^2 \geq \eta P$$

We can take $\gamma = \alpha^{K}$ and $\eta = e^{-\alpha^{-K}}$.

By Proposition 28 we can partition $\mathbb{Z}_N$ into translatio $P_1, P_2, \ldots, P_M$ of P such that for each i, $\exists\, r_i$ such that

$$\sum_i \left| \sum_{x \in P_i} f(x)\, \omega^{\lambda x^2 + r_i x} \right| \geq \eta N/2.$$

By Proposition 41 we can partition the P_i's into $P_{i,j}$'s of length $\geq |P|^{\frac{1}{18 \times 128}}$ such that

$$\sum_i \sum_j \left| \sum_{x \in P_{ij}} f(x) \right| \geq \eta N/4$$

By Proposition 44 we can assume that each P_{ij} is a $\mathbb{Z}$ A.P. and the average length in $\Omega(N^{\frac{1}{2 \times 18 \times 128}})$

and no P_i, is more than twice average length.

Re-label them as $Q_1, Q_2, \ldots, Q_M$ where

$$M = c N^{1 - \frac{\gamma}{2 \times 18 \times 128}}.$$

As $\sum_x f(x) = 0$ we have

$$\sum_i \left(\left| \sum_{x \in Q_i} f(x) \right| + \sum_{x \in Q_i} f(x) \right) \geq \eta N/4$$

The contribution from Q_i, $|Q_i| \leq \sqrt{\frac{N}{M}}$ is at most $M\sqrt{N/M} \ll \eta N$. So $\exists Q$, $|Q_i| \geq \sqrt{N/M}$ such that

$$\left| \sum_{x \in Q_i} f(x) \right| + \sum_{x \in Q_i} f(x) \geq \eta \frac{|Q_i|}{8}$$

and so

$$\sum_{x \in Q_i} f(x) \geq \eta \frac{|Q_i|}{16}.$$

Thus $\exists$ A.P. Q of length $\geq \sqrt{NM}$

$\geq N^{\frac{\gamma}{4 \times 18 \times 128}}$ such that

$$|A \cap Q| \geq \left(\delta + \frac{\eta}{16}\right)|Q|, \qquad \eta \geq e^{-\delta^{-C}}.$$

Proposition

Let $A_1, A_2, \dots, A_m \subseteq [N]$, $\alpha > 0$ and

$$\sum_{i=1}^{m} |A_i| \geq \alpha m N.$$

Then $\exists$ $B \subseteq [m]$, $|B| \geq \alpha^5 m/2$ such that for at least $\frac{9}{10}$ of the pairs $(i,j) \in B^2$,

$$|A_i \cap A_j| \geq \alpha^2 N/2.$$

Proof

Choose $x_1, x_2, \dots, x_5$ randomly from $[N]$ with replacement

$$B = \{ i : x_1, x_2, \dots, x_5 \in A_i \}.$$

$$\Pr(i \in B) = \left(\frac{|A_i|}{N}\right)^s$$

Jensen

$$E(|B|) = \sum_{i=1}^{m} \left(\frac{|A_i|}{N}\right)^s \geq m\left(\sum \frac{|A_i|}{mN}\right)^s \geq \alpha^s m.$$

So

$$E(|B|^2) \geq E(|B|)^2 \geq \alpha^{10} m^2$$

If $|A_i \cap A_j| \leq \alpha^2 N/2$ then

$$\Pr(i \in B, j \in B) \leq \left(\frac{\alpha^2}{2}\right)^s$$

So if $C = \{(i,j) \in B \times B : |A_i \cap A_j| < \alpha^2 N/2\}$

then $E(|C|) \leq \frac{\alpha^{10}}{2^5} m^2$.

So

$$E(|B|^2 - 16|C|) \geq \frac{\alpha^{10} m^2}{2}.$$

So $\exists B$ such that

(i) $$|B|^2 \geq \frac{\alpha^{10} m^2}{2}$$

(ii) $$|B|^2 \geq 16|C|.$$

$\square$

Proposition (Balog-Szemerédi-Gowers)

Let $A \subseteq$ abelian group. Suppose $\alpha > 0$

and $\left|\{(a,b,c,d) \in A^4 : a-b = c-d\}\right| \geq \alpha |A|^3$.

Then $\exists\, A' \subseteq A$ such that $|A'| \geq c|A|$

and $|A'-A'| \leq C|A|$ where c, C depend

only on α.

Proof

Let $|A| = n$ and $f(x) = \left|\{(a,b) : x = a-b\}\right|$.

Then

$$\sum_x f(x) = n^2 \,; \quad \sum_x f(x)^2 \geq \alpha n^3 \,; \quad f(x) \leq n$$

It follows that if

$$S = \{x : f(x) \geq \alpha n/2\} \text{ then}$$

$$|S| \geq \alpha n/2.$$

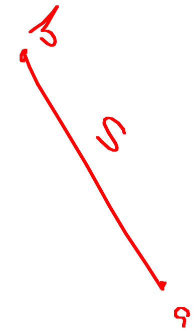
Note also that $x \in S \Rightarrow -x \in S$ $\quad f(x) = f(-x)$

$$\sum_x f(x) = n^2; \quad \sum f(x)^2 \geq \alpha n^3; \quad f(x) \leq n$$

$$\sum_x f(x)^2 \leq n \sum_{x \in S} f(x) + \frac{\alpha n}{2} \sum_{x \notin S} f(x) < \frac{\alpha n^3}{2} + \frac{\alpha n^3}{2}.$$

$$G = \text{graph}\left(A,\ E = \{ab :\ a - b \in S\}\right)$$

So
$$\sum_{a \in A} |N_G(a)| \geq \frac{\alpha^2 n^2}{4}$$



$$\geq \sum_{x \in S} f(x)$$

$\boxed{\text{sets are } N_G(a)\text{'s}}$

Applying Proposition 52 (with $N = m = n$) we find $\exists\, B \subseteq A$, $|B| \geq \frac{1}{2}\left(\frac{\alpha^2}{4}\right)^5 n$ such that $|N_G(a) \cap N_G(b)| \geq \frac{1}{2}\left(\frac{\alpha^2}{4}\right)^2 n$ for at least $9/10$ of $(a,b) \in B^2$.

$$H = \text{graph}\left(B, \left\{ab :\ |N_G(a) \cap N_G(b)| \geq \frac{\alpha^4 n}{32}\right\}\right)$$

Average degree in H is at least $\frac{9|B|}{10}$ and so at least $\frac{4}{5}|B|$ have degree $\geq \frac{4|B|}{5}$.

Let $A' = \{ a \in B : \deg_H(a) \geq \frac{4}{5}|B| \}$

Suppose $a, b \in A'$. There are at least $3|B|/5$ $c \in B$ such that $ac, bc \in E(H)$.

If $ac \in E(H)$ then $\exists \geq \frac{\alpha^4 n}{32}$ x such that $ax, cx \in E(G)$. Same for bc

---

If $ax \in E(G)$ then $\exists \geq \alpha n/2$ pairs u, v such that $u - v = x - a$.

Therefore there are at least

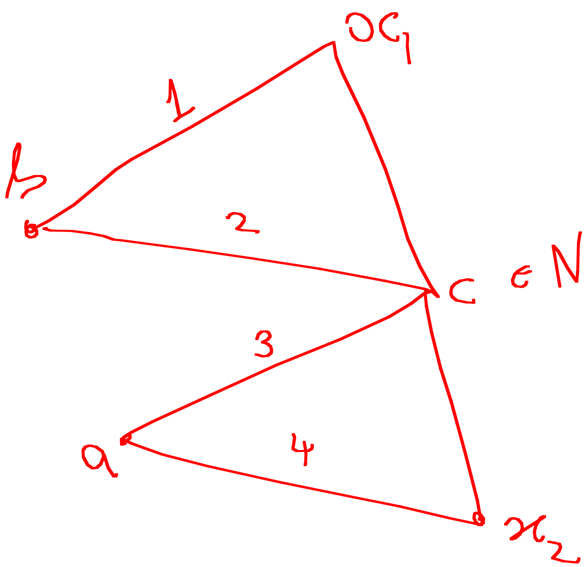
#choices c #choices x_1, x_2 #choices u_1-v_1 etc

$$\underbrace{\frac{3}{5}\frac{\alpha^{10}}{2^{11}}}_{|B|} n \times \left(\frac{\alpha^{4} n}{32}\right)^{2} \times \left(\frac{\alpha n}{2}\right)^{4}$$

octuples $(u_1, v_1, \dots, u_4, v_4) \in A^8$

such that

$$a + \underbrace{u_1 - v_1}_{x_2 - a} + \underbrace{u_2 - v_2}_{c - x_2} + \underbrace{u_3 - v_3}_{x_1 - c} + \underbrace{u_4 - v_4}_{b - x_1} = b$$



$c \in N_H(a,b)$ $\# c \geq \frac{3}{5}|B|$

If we choose different pair $(a', b') \in A' \times A'$ such that $b' - a' \neq b - a$ then we get a distinct set of octuple.

So

$$|A' - A'| \times \frac{3\alpha^{22}}{5 \times 2^{25}} n^7 \leq n^8$$

and

$$|A' - A'| \leq \frac{5 \times 2^{25}}{3 \times \alpha^{22}} n$$

Refined statement of Plünnecke Theorem

Suppose G is a commutative layered graph with vertex set $V = V_0 \cup V_1 \cup \cdots V_k$.

For $X, Y \subseteq V$ we let

$$\operatorname{im}(X, Y) := \{ y \in Y : \exists \text{ directed path from some } x \in X \text{ to } y \}$$

$$\mu(X, Y) = \min \left\{ \frac{|\operatorname{im}(Z, Y)|}{|Z|} : Z \subseteq X,\ Z \neq \emptyset \right\}$$

$$\mu_i = \mu(V_0, V_i)$$

Plünnecke: $\mu_j^{1/j}$ is decreasing.

Corollary: suppose $j < h$.

$$\exists X \subseteq V_0, \ X \neq \emptyset \quad \text{s.t.}$$

$$|\mathrm{im}(X, V_k)| \leq \left(\frac{|V_j|}{|V_0|}\right)^{h/j} |X|$$

X yields $\|$

$$\mu(V_0, V_h) \times |X| = \mu_h |X| \leq \mu_j^{h/j} |X|$$

<u>Corollary</u>: suppose $j < h$ and $A, B \leq G$.

$$\exists X \leq A,\ X \neq \emptyset : |X + hB| \leq \left(\frac{|A + jB|}{|A|} \right)^{h/j} |X|$$

Apply Cor. 62 to subgraph generated

$A, A+B, A+2B, \ldots, A+jB$.

Corollary: suppose $j < h$ and $A, B \subseteq G$.

Then

$$|hB| \leq \left(\frac{|A + jB|}{|A|}\right)^{h/j} |A|$$

$$\leq |X + hB| \leq \left(\frac{|A + jB|}{|A|}\right)^{h/j} |X|$$

from Cor. 63

$X \subseteq A$

In particular, if $B = -A$ then

$$|2A| \leq \left(\frac{|A - A|}{|A|}\right)^2 |A|$$

Proposition

Let $A \subseteq \mathbb{Z}^k$ with $|A| = m$ such that $|\{(a,b,c,d) \in A^4 : a-b = c-d\}| \geq \alpha m^3$.

Then $\exists$ G.A.P. Q of size $\leq Cm$ and dimension d such that $|A \cap Q| \geq cm$ where c, C, d depend only on α.

Proof

Proposition 55 shows that $\exists A' \subseteq A$ such that $|A'| \geq c_0 |A|$ and such that $|A' - A'| \leq c_1 |A'|$.

Then $|A' + A'| \leq c_1^2 |A'|$ and we can apply Freiman's Theorem.

Prop 64

Proof of Proposition 2.6

Let $\Gamma = \{ (b, \phi(b)) : b \in B \} \subseteq \mathbb{Z} \times \mathbb{Z}$.

Let $m = |B|$.

$A = \Gamma$ ✓ check condition

By Proposition 6.5, $\exists$ G.A.P. Q of size $\leq Cm$ and dimension d with $|\Gamma \cap Q| \geq cm$.

If $Q = Q_1 + Q_2 + \cdots + Q_d$ then we can assume $|Q_1| \geq (cm)^{1/d}$.

uses properness

Therefore Q can be partitioned into one dimensional ($\subseteq \mathbb{Z}^2$) A.P.'s of size at least $(cm)^{1/d}$. Therefore $\exists$ an A.P. $R \subseteq \mathbb{Z}^2$ such $|R \cap \Gamma| \geq \frac{c}{C}|R| \geq \frac{c}{C}(cm)^{1/d}$.

$$|R \cap \Gamma| \geq \frac{c}{C}|R| \geq \frac{c}{C}(cm)^{1/d}.$$

Suppose

$$R = \{(\underbrace{a_1 + kb_1}_{s}, \underbrace{a_2 + kb_2}_{\psi(s) = \phi(s) \text{ in } \Gamma \cap R}) : k \geq 0\}$$

$$\psi(s) = a_2 + \frac{s - a_1}{b_1} b_2$$

Reduce mod N.

Proof of Proposition 38

The Weyl Inequality (for quadratic monomials)

Let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ and $N \in \mathbb{N}$ with $N \geq 2$. If $\alpha \in \mathbb{R}$ with $|\alpha - a/q| \leq q^{-2}$ then

$$\left| \sum_{n=1}^{N} e^{2\pi i \alpha n^2} \right| \leq 20 \log N \left(N^{\frac{1}{2}} + q^{\frac{1}{2}} + N/q^{\frac{1}{2}} \right).$$

Lemma

Let $\alpha \in \mathbb{R}$. Then for all $N \in \mathbb{N}$

$$\left| \sum_{n=1}^{N} e^{2\pi i \alpha n} \right| \leq \min\left\{ N, \frac{1}{2\|\alpha\|} \right\}$$

where $\|\alpha\|$ = distance α to nearest integer.

Proof

If $\alpha = 0$ then the sum is N.

If $\alpha \neq 0$ then

$$\times \frac{e^{-\pi i \alpha n}}{e^{-\pi i \alpha n}}$$

$$\left| \sum_{n=1}^{N} e^{2\pi i \alpha n} \right| \leq \frac{\left|1 - e^{2\pi i \alpha N}\right|}{\left|1 - e^{2\pi i \alpha}\right|} = \frac{|\sin \pi \alpha N|}{|\sin \pi \alpha|} \leq \frac{1}{2\|\alpha\|}$$

$\square$

$$S = \sum_{n=1}^{N} e^{2\pi i P(n)}$$

P is a polynomial with real coefficients.

$$|S|^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} e^{2\pi i [P(m) - P(n)]}$$

$$= \sum_{n=1}^{N} \sum_{h=1-n}^{N-n} e^{2\pi i (P(n+h) - P(n))}$$

$h + n \in [1, N]$

$$= N + \sum_{h=1}^{N-1} \sum_{n=1}^{N-h} e^{2\pi i (P(n+h) - P(n))}$$

$$+ \sum_{h=1-N}^{-1} \sum_{n=1-h}^{N} e^{2\pi i (P(n+h) - P(n))}$$

$\sum_{-h=1}^{N-1} \quad \sum_{n+h=1}^{N-h}$

$P(n+h-h) - P(n+h)$

$$= N + 2 \operatorname{Re} \sum_{h=1}^{N-1} \sum_{n=1}^{N-h} e^{2\pi i (P(n+h) - P(n))}$$

$$\leq N + 2 \sum_{h=1}^{N-1} \left| \sum_{n=1}^{N-h} e^{2\pi i (P(n+h) - P(n))} \right|$$

Now let $P(n) = \alpha n^2$.

$$|S|^2 \leq N + 2\sum_{h=1}^{N-1}\left|\sum_{n=1}^{N-h} e^{2\pi i \alpha(2hn+\cancel{h^2})}\right|$$

$$\leq N + 2\sum_{h=1}^{N-1} \min\left\{N-h, \frac{1}{\|2\alpha h\|}\right\}$$

$$\leq N + 2\sum_{h=1}^{2N} \min\left\{N, \frac{1}{\|\alpha h\|}\right\}.$$

$$\left|\sum_{n=1}^{N} e^{2\pi i \alpha n^2}\right| \leq 20 \log N\left(N^{\frac{1}{2}} + q^{\frac{1}{2}} + N/q^{\frac{1}{2}}\right)$$

follows from:

$$\sum_{h=1}^{H=2N} \min\left\{N, \frac{1}{\|\alpha h\|}\right\} \leq 24 \log N\left(N + q + H + HN/q\right)$$

$$S = \left| \sum_{n=1}^{N} e^{2\pi i \alpha n^2} \right| \leq 20 \log N \left(N^{1/2} + q^{1/2} + N/q^{1/2} \right) = T$$

follows from:

$$\circledast \quad \sum_{h=1}^{H = 2N} \min\left\{ N, \frac{1}{\|\alpha h\|} \right\} \leq 24 \log N \left(N + q + H + HN/q \right)$$

$$S^2 \leq N + 48 \log N \left(N + q + 2N + 2N^2/q \right) \leq T^2$$

Proof of $\otimes$72: