On rainbow trees and cycles

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Michael Krivelevich
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Both theorems use the (lop-sided) local lemma.
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**Theorem**

There exists an absolute constant $c > 0$ such that if an edge colouring of $K_n$ is $cn$-bounded then there exist rainbow cycles of all sizes $3 \leq k \leq n$. 
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We see immediately from AFR that if $b \leq n/128$ then every $b$-bounded coloring of $K_n$ contains rainbow cycles of lengths $n/2 \leq k \leq n$. Indeed every set of $n/2 \leq k \leq n$ vertices contains a spanning rainbow cycle.
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For smaller $k$ we use the following: If $c > 0$ and an edge colouring of $K_n$ is $cn$-bounded, then there exists a set $S \subseteq [n]$ such that $|S| = N = n/2$ and the induced colouring of the edges of $S$ is $c'N$-bounded where $c' = c(1 + 1/(\ln n)^2)$. 
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To prove this, we take a random $n/2$ set $S$.

To complete the theorem, we take $c$ sufficiently small and we apply this $\sim \log_2 n$ times until we have shown the existence of rainbow cycles of length $N \leq k \leq n$ where $cN \leq 1$ and a set of $N$ vertices for which the edge colouring is $cN$ bounded.
Rainbow Trees

**Theorem**

Given a real constant $\epsilon > 0$ and a positive integer $\Delta$, there exists a constant $c = c(\epsilon, \Delta)$ such that if $n \geq (1 - \epsilon)\Delta$ and an edge colouring of $K_n$ is $cn$-bounded, then it contains a rainbow copy of every tree $T$ with at most $(1 - \epsilon)n$ vertices and maximum degree $\Delta$. 
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Conjecture: There is a constant $c = c(\Delta)$ such that every $cn$-bounded edge colouring of $K_n$ contains a rainbow copy of every spanning tree of $K_n$ which has maximum degree at most $\Delta$. 
Our main tool is a theorem of Alon, Krivelevich and Sudakov:

Suppose that $\Delta \geq 2$ and $0 < \epsilon < 1/2$. Let $H$ be a graph on $N$ vertices with minimum degree $\delta_H$ and maximum degree $\Delta_H$.

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- \( N \) is sufficiently large.
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- \( H \) has sufficiently good expansion.

Then \( H \) contains a copy of every tree with \( \leq (1 - \epsilon)N \) vertices and maximum degree \( \leq \Delta \).
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Existence of rainbow trees has now been demonstrated.
Using the (lop-sided) local lemma one can also prove:

Let $T$ be an arbitrary rooted tree with $\nu$ vertices.

Let $T_1, T_2, \ldots, T_\nu$ be copies of $T$ with roots $x_1, \ldots, x_\nu$.

Run a path through $x_1, \ldots, x_\nu$ to create the tree $T(\nu)$.

There exists an absolute constant $c > 0$ such that if an edge colouring of $K_n$ is $cn$-bounded then there exists a rainbow copy of every possible $T(\nu)$.
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- Show that there exists a constant $c > 0$ such in every $cn$-bounded colouring of $K_n$ there are an exponential number of rainbow Hamilton cycles.
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- Construct a polynomial time algorithm to find a rainbow Hamilton cycle in a $cn$-bounded coloring of $K_n$. 
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- Construct a polynomial time algorithm to find a random rainbow Hamilton cycle in a $cn$-bounded coloring of $K_n$. 
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- Show that there exists $c = c(\Delta)$ so that every $cn$-bounded colouring of $K_n$ contains a rainbow copy of every tree with $n$ vertices and with maximum degree $\leq \Delta$.
- Show that there exists a constant $c > 0$ such in every $cn$-bounded colouring of $K_n$ there are an exponential number of rainbow Hamilton cycles.
- Construct a polynomial time algorithm to find a rainbow Hamilton cycle in a $cn$-bounded coloring of $K_n$.
- Construct a polynomial time algorithm to find a random rainbow Hamilton cycle in a $cn$-bounded coloring of $K_n$.
- For what values of $c, p$ does a $cnp$ bounded coloring of $G_{n,p}$ contain a rainbow Hamilton cycle whp?
THANK YOU