Packing Hamilton Cycles in $\varepsilon$-Regular Graphs

Alan Frieze
Michael Krivelevich
General question:

What is the maximum number of edge disjoint Hamilton cycles that can be found in a graph $G$?
Background: Random graphs

Let a graph $G$ have property $H_k$ if it contains a collection of $b_k = 2^c$ edge disjoint Hamilton cycles plus a further edge disjoint perfect matching if $k$ is odd.

Clearly, $G = 2^H_k$ if $k > (G)$.

Consider the graph process $G_0; G_1; \ldots; G_t$ where $G_0 = (\{n\}; n)$ and $G_i + 1$ is obtained from $G_i$ by adding a random edge. Let $t_k = \min\{t : (G_t) = k\}$ and $h_k = \min\{t : G_t \in H_k\}$. 

- p.3
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Let $t_k = \min \{ t : \delta(G_t) = k \}$ and $h_k = \min \{ t : G_t \in \mathcal{H}_k \}$. 
Theorem Bollobás and Frieze(1985)
If $k = o(\ln \ln n)$ then whp $t_k = h_k$. 

Conjecture: Whp $G_{t^2 H}(G_{t^2})$ for $1 < t^2 n$. 

Question: What is $H(G_{t^2})$?
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If \( k = o(\ln \ln n) \) then \( \text{whp} \ t_k = h_k \).

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Question: What is $\pi_H(G_{n,1/2})$?

Theorem Whp $\pi_H(G_{n,1/2}) \geq \frac{n}{4} - O(n^{5/6} \ln n)$.
\( \varepsilon \)-regular graphs
Let $G_{n, \alpha, \varepsilon}$ denote the set of graphs $G$ on vertex set $[n]$ which have the following properties:

**P1** $\delta(G) \geq \alpha n$.

**P2** If $S, T$ are disjoint subsets of $[n]$ and $|S|, |T| \geq \varepsilon n$ then 
\[
\left| \frac{e_G(S,T)}{|S||T|} - \alpha \right| \leq \varepsilon,
\] where $e_G(S, T)$ is the number of $S - T$ edges in $G$. 

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**Nb:** Whp $G_{n,p} \in \mathcal{G}(n, p - O(n^{-1/2} \ln n), O(n^{-1/3} \ln n))$, for constant $p$. 

**$\varepsilon$-regular graphs**
Theorem  Suppose that $\alpha$ is constant and

$$10 \left( \frac{\ln n}{n} \right)^{1/6} \leq \varepsilon \ll \alpha.$$ 

If $G \in G_{n,\alpha,\varepsilon}$ then $G$ contains $(\frac{\alpha}{2} - 3\varepsilon)n$ edge disjoint Hamilton cycles.
Bipartite Graphs: Make analogous definition $B_{n,\alpha,\varepsilon}$ and obtain analogous result:

Directed Graphs: Make analogous definition $D_{n,\alpha,\varepsilon}$ and obtain analogous result:
Hamilton cycle game

Maker and Breaker alternately choose edges from the complete graph $K_n$. Maker aims to choose as many edge disjoint Hamilton cycles as possible among his edges. Theorem: Maker can choose $n^4\Omega(n^2)$ edge disjoint Hamilton cycles. Improves a result of Lu who showed $n^{16}$ could be chosen. Played on $K_n$, we have the same result. Played on $D_n$, Maker can choose $n^2\Omega(n)$ edge disjoint directed Hamilton cycles.
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**Theorem** Maker can choose $\frac{n}{4} - o(n)$ edge disjoint Hamilton cycles.
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Acknowledgements
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Noga Alon for suggesting the use of some approximations to 0-1 permanents in regards to showing the existence of 2-factors with few cycles.
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Oleg Pikhurko for suggesting that the Erdős-Selfridge method, combined with our theorem on packing Hamilton cycles yields the result on Hamilton cycle games.
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$F$ can be partitioned into $s$ 2-factors, $F_1, \ldots, F_s$ each having $o(n)$ cycles.
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Each $F_i$ can be converted into a Hamilton cycle preserving disjointness and with the help of $\Gamma$. 
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Let $Q(S, T)$ be the number of odd components of the graph $G_U$ induced by $U$. A component $C$ of $G_U$ is odd if $r|C| + e_{G_1}(C, T)$ is odd.
$G_1$ contains an $r$-factor iff for every partition of $[n]$ into $S, T, U$ we have $R(S, T) \geq Q(S, T)$. 
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If $|T| \leq (1 - \frac{3\varepsilon}{\alpha - \varepsilon})n$ then $R(S, T) \geq \varepsilon n|T| - n \geq \varepsilon^2 n^2 - n \gg n$ and $Q(S, T) \leq |U| < n$. 
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If $|T| > (1 - \frac{3\varepsilon}{\alpha - \varepsilon})n$ then $|S| < \frac{3\varepsilon}{\alpha - \varepsilon} n$ and $R(S, T) \geq \varepsilon n|T| - n - 3\varepsilon n|S| \geq \varepsilon n (1 - \frac{3\varepsilon}{\alpha - \varepsilon}) n - n - \frac{9\varepsilon^2}{\alpha - \varepsilon} n^2 \gg n > |U|$.
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Another 4 equally simple cases finish the proof.
Extracting 2-factors

**Lemma** $H$ is a $2d$-regular graph on vertex set $[n]$, where $d \geq \varepsilon n$. $H$ contains a 2-factor with at most $10\varepsilon^{-1}(n \ln n)^{1/2}$ cycles.
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**Lemma** $H$ is a $2d$-regular graph on vertex set $[n]$, where $d \geq \varepsilon n$. $H$ contains a 2-factor with at most $10\varepsilon^{-1}(n \ln n)^{1/2}$ cycles.

**Proof**
For simplicity we just prove $O(n/\ln n)$ cycles.
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$B$ is the bipartite graph with vertex set $[n] + [n]$ where $(x, y)$ is an edge of $B$ iff $(x, y)$ is an arc of $\vec{H}$.
Extracting 2-factors

Lemma \( H \) is a \( 2d \)-regular graph on vertex set \([n]\), where \( d \geq \varepsilon n \). \( H \) contains a 2-factor with at most \( 10\varepsilon^{-1}(n \ln n)^{1/2} \) cycles.

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Orient \( H \) so that vertices of \( \vec{H} \) have in-degree=out-degree \( d \)

\( B \) is the bipartite graph with vertex set \([n] + [n]\) where \((x, y)\) is an edge of \( B \) iff \((x, y)\) is an arc of \( \vec{H} \)

A perfect matching \( M \) of \( B \) yields a unique 2-factor \( C_M \) of \( H \).
$X$ is the number of perfect matchings of $B$. 

Van der Waerden's conjecture. – p.15
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Van der Waerden’s conjecture.

\( X_{k,\ell} \) is the number of perfect matchings \( M \) of \( B \) such that \( C_M \) contains at least \( k \) cycles of length \( \ell \).

\[
X_{k,\ell} \leq \binom{n}{k} d^{k(\ell-1)} \ell^{-k} (d!)^{(n-k\ell)/d}.
\]

Minc conjecture
\[ X^{-1}X_{k,\ell} \leq O(n) \left( \frac{ne^{\ell}}{dk\ell} \right)^k. \]
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Put \( k = n^{3/4} \) and \( \ell_0 = \frac{1}{2} \ln n \). Then

\[ X^{-1} \sum_{\ell=3}^{\ell_0} X_{k,\ell} < 1. \]
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Put \( k = n^{3/4} \) and \( \ell_0 = \frac{1}{2} \ln n \). Then

\[ X^{-1} \sum_{\ell=3}^{\ell_0} X_{k,\ell} < 1. \]

So there exists a 2-factor with at most

\[ n^{3/4}\ell_0 + \frac{n}{\ell_0} \leq 2\frac{n}{\ell_0} \]

cycles.
Transforming 2-factors to Hamilton cycles

Assume inductively that for some $i \geq 0$ we have created edge-disjoint Hamilton cycles $H_1, H_2, \ldots, H_i$ which are edge-disjoint from $F_{i+1}, \ldots, F_{s-\varepsilon n}$. 
Transforming 2-factors to Hamilton cycles

Assume inductively that for some $i \geq 0$ we have created edge disjoint Hamilton cycles $H_1, H_2, \ldots, H_i$ which are edge-disjoint from $F_{i+1}, \ldots, F_{s-\varepsilon n}$.

Assume further that $|H_j \setminus F_j| \leq 6\ell_0$ for $1 \leq j \leq i$. 
Continue until path cannot be extended in this way
At least $2\varepsilon n$ choices for $x$ and for $y$
Edge of $\Gamma_1$
We need at most 3 edges of $\Gamma_1$ to reduce the number of cycles by one.

At most $6n^2/\ell_0 \ll \varepsilon^2 n^2$ are needed altogether to create $\sim \alpha n/2$ disjoint Hamilton cycles.
Bipartite graphs: very similar.

Only need to observe that the path lengths are odd in the Hamilton cycle stage.
Digraphs:
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We show that Maker can build a graph in $G_{n; 1/2 \ldots 1/4}$.

$F$ is the hypergraph with hyper-edges $A_1; A_2; \ldots; A_k$ and $B_1; B_2; \ldots; B_j$.

$A_1; A_2; \ldots; A_k$ is a collection of $\binom{n}{M}$-sets where $M = (\log n)^2$ and $M = \frac{2}{\log n}$. For $1 \le i \le n$, every $d(1 = 2 + \ldots + 4) \subset$ subset of the edges incident with $i$ contains at least one of $A_i$ and $B_j$.

$B_1; B_2; \ldots; B_j$ enumerate following edge sets: Choose disjoint subsets $S; T$ of $\binom{n}{\log n}$, of size at least $\frac{n}{4}$. Choose subset $B$ of $S \cup T$ edges with $|B| = d|S|\frac{|T|}{(1 + \ldots + 4)} = 2e$. 

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$\mathcal{F}$ is the hypergraph with hyper-edges $A_1, A_2, \ldots, A_\mu, B_1, B_2, \ldots, B_\nu$. 
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$A_1, A_2, \ldots, A_\mu$ is a collection of $\mu = nM$ $\ell$-sets where $\ell = (\log n)^2$ and $M = 2^\ell / n^2$. For $1 \leq i \leq n$, every $\lceil (1/2 + \varepsilon/4)n \rceil$ subset of the edges incident with $i$ contains at least one of $A_{(i-1)M+j}$, $1 \leq j \leq M$. 
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$B_1, B_2, \ldots, B_\nu$ enumerate the following edge sets: Choose disjoint subsets $S, T$ of $[n]$, of size at least $\varepsilon n / 4$. Choose subset $B$ of $S - T$ edges with $|B| = \lceil |S||T|(1 + \varepsilon/4)/2 \rceil$. 
Maker has only to ensure that the choices $E_M, E_B$ of Maker, Breaker respectively are a 2-colouring of $F$. 
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**Lemma** If the hyper-edges of a hypergraph $\mathcal{F}$ satisfy
$$\sum_{X \in \mathcal{F}} 2^{1-|X|} < 1$$
then Maker can force a 2-colouring of $\mathcal{F}$.
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**Lemma** If the hyper-edges of a hypergraph \( \mathcal{F} \) satisfy
\[
\sum_{X \in \mathcal{F}} 2^{1-|X|} < 1
\]
then Maker can force a 2-colouring of \( \mathcal{F} \).

The sum in the lemma is the expected number of mono-coloured hyper-edges in a random 2-colouring of \( \mathcal{F} \).
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For $X \in \mathcal{F}$, $\delta_{X,M}, \delta_{X,B}$ are the 0-1 indicators for $X \cap C_M \neq \emptyset$, $X \cap C_B \neq \emptyset$ respectively.

$\delta_X = \delta_{X,M} + \delta_{X,B}$. 
Potential function

\[ \Phi = \sum_{\substack{X \in \mathcal{F} \\ \delta_X \leq 1}} 2^{-|X \cap R| + 1 - \delta_X}. \]
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\]

This is \(< 1\) initially, by assumption and if it is \(< 1\) at the end when \(R = \emptyset\) then Maker has succeeded.
Suppose that in some round, Maker chooses edge $a$ and Breaker chooses edge $b$. $\Phi'$ is the new value of $\Phi$. 
Suppose that in some round, Maker chooses edge $a$ and Breaker chooses edge $b$. $\Phi'$ is the new value of $\Phi$.

$$
\Phi' - \Phi = - \sum_{\substack{a,b \in X \\delta_X = 0}} 2^{|X \cap R|} - \sum_{\substack{a \in X \\delta_{X,B} = 1}} 2^{-|X \cap R|} - \sum_{\substack{b \in X \\delta_{X,M} = 1}} 2^{-|X \cap R|} \\
+ \sum_{\substack{a \in X, b \notin X \\delta_{X,M} = 1}} 2^{-|X \cap R|} + \sum_{\substack{a \notin X, b \in X \\delta_{X,B} = 1}} 2^{-|X \cap R|}
$$
\[
\leq - \left( \sum_{a \in X} 2^{-|X \cap R|} - \sum_{a \in X} 2^{-|X \cap R|} \right) \delta_{X,B = 1} \delta_{X,M = 1} \\
+ \left( \sum_{b \in X} 2^{-|X \cap R|} - \sum_{b \in X} 2^{-|X \cap R|} \right) \delta_{X,B = 1} \delta_{X,M = 1}
\]
\[
\leq - \left( \sum_{a \in X \atop \delta_{X,B} = 1} 2^{-|X \cap R|} - \sum_{a \in X \atop \delta_{X,M} = 1} 2^{-|X \cap R|} \right) \\
+ \left( \sum_{b \in X \atop \delta_{X,B} = 1} 2^{-|X \cap R|} - \sum_{b \in X \atop \delta_{X,M} = 1} 2^{-|X \cap R|} \right)
\]

which is non-positive if Maker chooses \( a \) to maximise

\[
\sum_{a \in X \atop \delta_{X,B} = 1} 2^{-|X \cap R|} - \sum_{a \in X \atop \delta_{X,M} = 1} 2^{-|X \cap R|}
\]