Mathematical Models of the WWW and related networks

Alan Frieze

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- Peer to Peer Networks

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Vertex set $\{1, 2, \ldots, n\}$, $N = \binom{n}{2}$
Each of the $\binom{N}{m}$ graphs with m edges is equally likely.

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In many real world cases

$$n_k \sim \frac{A}{k^\alpha} n \qquad \text{power law}$$

for some constants A, α e.g. Faloutsos,Faloutsos,Faloutsos

Fixed (expected) degree sequence models.

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Let $\Theta = \sum_i d_i(d_i - 2)$.

$\Theta < 0$ implies all components of G are small **whp**

$\Theta > 0$ implies G contains a giant component (size $\Omega(n)$) **whp**

Digraphs with a fixed degree sequence: Cooper and Frieze

Consider a random digraph D with n vertices and $\ell_{i,j}$ vertices of in-degree i and out-degree j.

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$d = \sum_{i,j} ij\ell_{i,j}/(\theta n)$ is the expected out-degree of a randomly chosen arc.

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$\theta < 0$ implies all strong components of G are small

$\theta > 0$ implies G has a giant strong component S (size $\Omega(n)$)

More on $d > 1$.

Let L^+ be the set of vertices with a giant “fan-out” and L^- be the set of vertices with a giant “fan-in”. Then **whp**

$$S = L^+ \cap L^-$$

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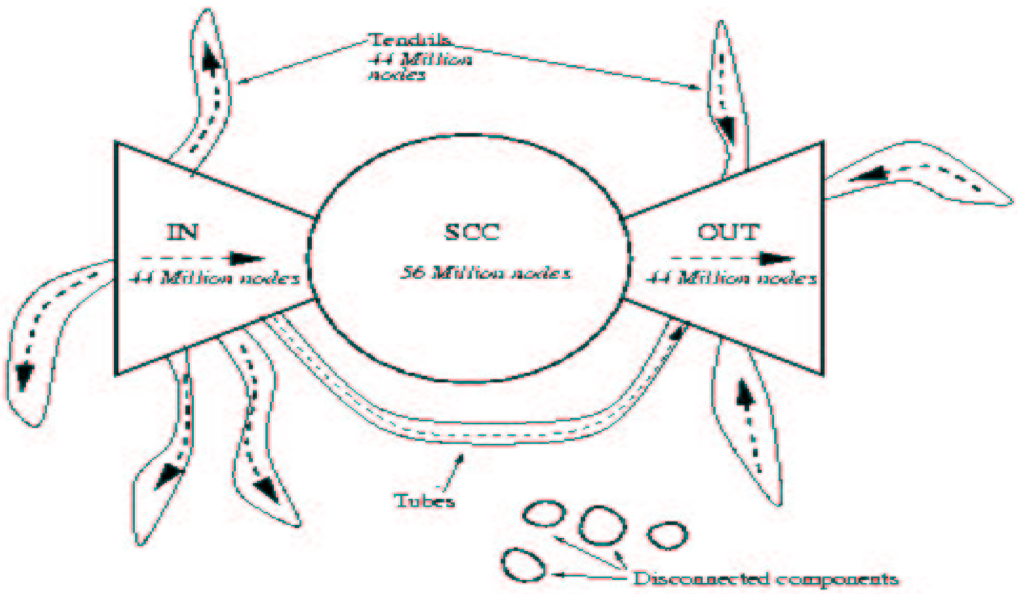

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Expected degree model
Fix $a_1, a_2, \ldots, a_n$. Let $A = a_1 + \cdots + a_n$. Then put an edge between i and j with probability $a_i a_j / A$.

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Let $\lambda_1 \geq \lambda_2 \geq \cdots$ be the largest eigenvalues of the adjacency matrix of the graph produced.

Suppose that $a_i = \frac{a_1}{i^\alpha}, 1/2 < \alpha < 1$ for $1 \leq i \leq n^\beta$, β sufficiently small.

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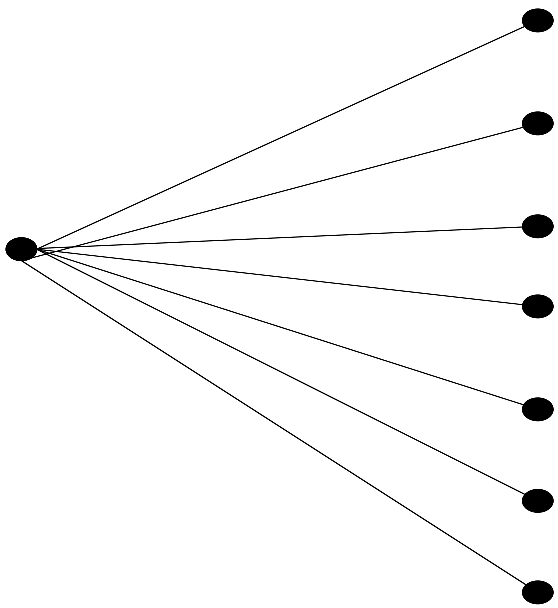

Star of degree d:
largest eigenvalue $d^{1/2}$

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Preferential attachment also used in models by Yule 1925 and Simon 1955.

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$$d_k \sim \frac{2m(m+1)}{(k+2)(k+1)k}t \qquad for\ k \geq m.$$

Bollobás,Riordan,Spencer,Tusnady

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Expected degree of vertex v_s at time t is $\sim m\sqrt{t/s}$.
If $i < j$,

$$\mathbf{Pr}(Edge\ (v_i, v_j)\ exists) \leq \omega \cdot \frac{\sqrt{j/i}}{2mj} \leq \frac{\omega}{\sqrt{ij}}$$

where $\omega \to \infty$ slowly.

Diameter: Bollobás,Riordan.

$$Diameter \sim \frac{\log t}{\log\log t}$$

Let $k = (1-\epsilon)\log t / \log\log t$

$$\begin{aligned}
\mathbf{Pr}(\exists Path\ length\ k, t \to t-1) &\leq \sum_{t_0=t,\ldots,t_k=t-1} \prod_{i=1}^{k} \frac{\omega}{\sqrt{t_{i-1}t_i}} \\
&\leq \omega^k \frac{1}{t(t-1)} \left(\sum_{i=1}^{k} \frac{1}{i} \right)^k \\
&= o(1).
\end{aligned}$$

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Upper bound is harder

Copying Model

Communities: A large dense bipartite sub-graph of the WWW indicates a “community”. Experiments indicate a larger number of communities than you would get say from the simple model PAM.

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The edges of these bipartite cliques are oriented the same way.

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2. With probability $1 - \alpha$ we create edge (v_t, y) where y is the ith choice of u.

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This contrasts with PAM which has $O(1)$ in expectation.

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Robustness: Suppose we build our PAM graph and then delete the first ct vertices. What remains has a giant ($\Omega(t)$ size) component iff

$$c < \frac{m-1}{m+1}$$

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What about deletions during the growing phase?

Random deletion in a scale free random graph

We consider a model where edges are added using preferential attachment and vertices/edges are deleted randomly. Flaxman, Frieze, Vera

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Same model considered by Chung and Lu.

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- D Probability $\alpha - \alpha_1$: Add m random edges with endpoints chosen independently as in (-3).

We skip over details of what to do if there are no vertices to delete or what to do with multiple edges etc.

$D_k(t)$ is the number of vertices of degree k in G_t and $\overline{D}_k(t) = \mathbf{E}(D_k(t))$.

$$\beta = \frac{2(\alpha - \alpha_0)}{3\alpha - 1 - \alpha_1 - \alpha_0}$$

Theorem

Under natural restrictions on the parameters, there exists a constant $C = C(m, \alpha, \alpha_0, \alpha_1)$ such that for $k \geq 1$,

$$\left| \frac{\overline{D}_k(t)}{t} - Ck^{-1-\beta} \right| = O\left(t^{-\epsilon}\right) + O\left(k^{-2-\beta}\right).$$

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$$\overline{D}_k(t+1) = \overline{D}_k(t) +$$

$$(2\alpha - \alpha_1) m \mathsf{E}\left(-\frac{k D_k(t)}{2e_t} + \frac{(k-1) D_{k-1}(t)}{2e_t} \;\middle|\; e_t > 0 \right) \mathbf{Pr}(e_t > 0)$$

$$+ (1-\alpha)(k+1) \mathsf{E}\left(\frac{D_{k+1}(t) - D_k(t)}{v_t} \;\middle|\; e_t > 0 \right) \mathbf{Pr}(e_t > 0)$$

$$+ \alpha_1 1_{k=m} + \text{ error terms}.$$

Let

$$\nu = \alpha + \alpha_0 + \alpha_1 - 1 > 0 \text{ and } \eta = \frac{m(\alpha - \alpha_0)\nu}{1 + \alpha_1 - \alpha - \alpha_0}.$$

We show

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$$|v_t - \nu t| \leq t^{1/2} \log t, \qquad \mathbf{qs}.$$

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We used Chebychef to handle e_t. Chung and Lu modify Azuma's inequality, avoids constraint on edge deletion prob.

As a consequence we can write

$$\overline{D}_k(t+1) = \overline{D}_k(t) + (A_2(k+1) + B_2)\frac{\overline{D}_{k+1}(t)}{t} + (A_1 k + B_1 + 1)\frac{\overline{D}_k(t)}{t} + (A_0(k-1) + B_0)\frac{\overline{D}_{k-1}(t)}{t} + \alpha_1 1_{k=m} + O(t^{-\epsilon}).$$

$$A_2 = \frac{1-\alpha-\alpha_0}{\nu} + \frac{m\alpha_0}{\eta} \qquad B_2 = 0$$

$$A_1 = -\frac{(2\alpha-\alpha_1+2\alpha_0)m}{2\eta} - \frac{1-\alpha-\alpha_0}{\nu} \qquad B_1 = -1 - \frac{1-\alpha-\alpha_0}{\nu}$$

$$A_0 = \frac{(2\alpha-\alpha_1)m}{2\eta} \qquad B_0 = 0$$

Assume $\overline{D}_k(t) \sim d_k t$. $d_{-1} = 0$ and for $k \geq -1$,

$$(A_2(k+2) + B_2)d_{k+2} + (A_1(k+1) + B_1)d_{k+1} + (A_0 k + B_0)d_k = -\alpha_1 1_{k=m-1}. \qquad \text{(-2)}$$

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We first tackle homogeneous equation: for $k \geq 1$

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We use Laplace's method and make the substitution

$$e_k = \int_{t=a}^{t=b} t^{k-1} v(t) dt$$

for $a, b, v(t)$ to be determined.

Integrating by parts

$$ke_k = [t^k v(t)]_a^b - \int_{t=a}^{t=b} t^k v'(t) dt$$

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Let

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Substituting gives

$$[t^k\phi_1(t)v(t)]_a^b - \int_a^b t^k\phi_1(t)v'(t)dt + \int_a^b t^{k-1}\phi_0(t)v(t)dt = 0.$$

So, $v(t)$ will give a solution to the homogeneous equation if

$$[t^k v(t)\phi_1(t)]_a^b = 0 \qquad \text{and} \qquad \frac{v'(t)}{v(t)} = \frac{\phi_0(t)}{t\phi_1(t)}.$$

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The differential equation is homogeneous and can be integrated to give,

$$v(t) = C_0(t-1)^{\beta}\,(t-A)^{-\beta}$$

where $A > 1$ and $C_0 \neq 0$.

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Substituting we get the following solution to the homogeneous equation: valid for $k \geq 1$,

$$u_1(k) = \int_0^1 t^{k-1} \left(\frac{1-t}{1-\gamma t} \right)^\beta dt$$

where

$$\gamma < 1.$$

$$
\begin{aligned}
u_1(k) &= \int_0^1 t^{k-1}(1-t)^\beta \frac{1}{(1-\gamma t)^\beta} dt \\
&= \int_0^1 t^{k-1}(1-t)^\beta \sum_{j=0}^{\infty} \binom{\beta+j-1}{j} (\gamma t)^j dt \\
&= \sum_{j=0}^{\infty} \binom{\beta+j-1}{j} \gamma^j \int_0^1 t^{k+j-1}(1-t)^\beta dt \\
&= \sum_{j=0}^{\infty} \binom{\beta+j-1}{j} \gamma^j \frac{\Gamma(k+j)\Gamma(\beta+1)}{\Gamma(k+j+\beta+1)} \\
&= \sum_{j=0}^{\infty} \gamma^j \frac{\Gamma(\beta+j)}{\Gamma(j+1)\Gamma(\beta)} \frac{\Gamma(k+j)\Gamma(\beta+1)}{\Gamma(k+j+\beta+1)}
\end{aligned}
$$

assuming k is large, using Stirling for $\Gamma(k+j), \Gamma(k+j+\beta+1)$, we get

$$
\begin{aligned}
&= (1+O(k^{-1}))\beta \sum_{j=0}^{\infty} \gamma^j \frac{\Gamma(j+\beta)}{\Gamma(j+1)} (k+\beta+j)^{-\beta-1} \\
&= (1+O(k^{-1}))C_1 k^{-1-\beta}
\end{aligned}
$$

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We need to consider the non-homogeneous equation.

$d_{-1} = 0$ and for $k \geq -1$,

$$(A_2(k+2) + B_2)d_{k+2} + (A_1(k+1) + B_1)d_{k+1} + (A_0 k + B_0)d_k = -\alpha_1 1_{k=m-1}. \qquad \text{(-2)}$$

We show there is a solution such that

$$d_k = C u_1(k)$$

for $k \geq m$.

Adversarial deletion in a scale free random graph:
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On the other hand, we use their idea of coupling with $G_{n,p}$.

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Step t: Add m random edges from

$$X_t \text{ to } Y_1, Y_2, \ldots, Y_m \in \{X_1, X_2, \ldots, X_{t-1}\}$$

where

$$|Y_i - X_t| \leq r = n^{\epsilon - 1/2}$$

and the Y_i's are chosen via preferential attachment.

Theorem

- If m is a sufficiently large constant then there exists a constant $c > 0$ such that

$$d_k(n) \sim \frac{cn}{k(k+1)(k+2)}$$

where $d_k(n)$ is the number of vertices of degree k at time n.

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- If $m \geq K \log n$ and K is sufficiently large then **whp** G_n is connected.

Heuristically Optimized Trade-offs

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Then at step $i+1$, X_{i+1} is joined by an edge to the vertex X_j which minimises

$$\alpha|X_{i+1} - X_j| + h_j$$

where h_j is the number of edges from X_j to X_1 in T_i.

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If $\alpha \ge 4$ and $\alpha = o(n^{1/2})$ then **E**$(\rho_{\ge k}) \ge c(k/n)^{\beta}$.

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Some of these results are from Berger,Bollobás,Borgs,Chayes,Riordan:
Also Berger,Borgs,Chayes,D'Souza,Kleinberg

Crawling on web graphs

Cooper and Frieze

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Sequence of random graphs $G(t) : G(t) = G(t-1)$ plus vertex t and m random edges $\{t, v_i\}, i = 1, 2, \ldots, m$.

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Model 2 The vertices $v_1, v_2, \ldots, v_m$ are chosen proportional to their degree after step $t-1$.

Spider $\mathcal{S}$ sits at vertex X_{t-1} of $G(t-1)$. After the addition of vertex t, and before step $t+1$, spider makes a random walk of length ℓ

$\nu_{\ell,m}(t)$ is the expected number of vertices **not** visited by $\mathcal{S}$ at the end of step t.

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Theorem

In either model, if m is sufficiently large then,

$$\nu_{\ell,m}(t) \sim \mathbf{E} \sum_{s=1}^{t} \prod_{\tau=s}^{t} \left(1 - \frac{d(s,\tau)}{2m\tau}\right)^{\ell}$$

where $d(s,\tau)$ denotes the degree of s in $G(\tau)$.

Error in expression of order m^{-1} omitted.

Let

$$\eta_\ell = \lim_{m\to\infty} \lim_{t\to\infty} \frac{\mathbf{E}\nu_{\ell,m}(t)}{t}.$$

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(a) For Model 1,

$$\eta_\ell = \sqrt{\frac{2}{\ell}} e^{(\ell+2)^2/(4\ell)} \int_{(\ell+2)/\sqrt{2\ell}}^{\infty} e^{-y^2/2}\, dy$$

where $\Psi(x)$ is the standardized Normal cumulate for the interval $(-\infty, x]$.
In particular, $\eta_1 = 0.57\cdots$ and $\eta_\ell \sim 2/\ell$ as $\ell \to \infty$.

Let

$$\eta_\ell = \lim_{m\to\infty} \lim_{t\to\infty} \frac{\mathbf{E}\nu_{\ell,m}(t)}{t}.$$

(b) For Model 2

$$\eta_\ell = e^\ell 2\ell^2 \int_\ell^\infty y^{-3} e^{-y}\, dy.$$

In particular, $\eta_1 = 0.59\cdots$. and $\eta_\ell \sim 2/\ell$ as $\ell \to \infty$.

Further work

- Try other models of a random web-graph.
- Try non-uniform random walks.
- Prove concentration of the number of vertices visited.

The proof technique is robust enough to handle other models and walks, once one has established rapid mixing. The calculations for various non-uniform walks can get tedious.
It should be possible to estimate the variance of the number of unvisited vertices and apply Chebychef. Stronger concentration seems more challenging.

Cover time of preferential attachment graph

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$(1 - o(1))n \log n \leq C_G \leq (1 + o(1))\frac{4}{27}n^3$: Feige

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Cooper and Frieze If G is preferential attachment graph then **whp**

$$C_G \sim \frac{2m}{m-1} n \log n$$

Effect of search engines: Chakrabarti, Frieze, Vera

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At time step t we add vertex v_t and m randomly chosen edges to $\{u_1, \ldots, u_m\}$. For each i, u_i is chosen preferentially from

- With probability p we choose u_i from N vertices of largest degree..
- With probability $q = 1 - p$ we choose u_i from all vertices.

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Supposed to model surfer who obtains links from first N given by search engine.

Theorem

(a) For $i \leq N$ there exists constant $\alpha_i > 0$ such that

$$\mathbf{E}\left[\deg_n(x_i)\right] = \alpha_i n + O\left(n^{1/2}\right)$$

(b) There is an absolute constant A_1 such that for every

$$k \geq m, \bar{d}_k(n) = (1 + o(1))\frac{A_1 n}{k^{1+2/(1-p)}}.$$

where $\bar{d}_k(n)$ is the expected number of vertices of degree k outside the N largest vertices.