Probabilistic Method

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Rough outline

The basic Probabilistic method can be described as follows:

In order to prove the existence of a combinatorial structure with certain properties, we construct an appropriate probability space and show that a randomly chosen element in this space has desired properties with positive probability.
Ramsey theory

*Of three ordinary adults,*
*two must have the same sex.*

D.J. Kleitman

Ramsey Theory refers to a large body of deep results in mathematics with underlying philosophy: *in large systems complete disorder is impossible!*

*Theorem:* (Ramsey 1930)

∀ k, l there exists N(k, l) such that any two-coloring of the edges of complete graph on N vertices contains either/or

• Red complete graph of size k

• Green complete graph of size l
**Ramsey numbers**

**Definition:**

$R(k, l)$ is the minimal $N$ so that every red–green edge coloring of $K_N$ contains

- Red complete graph of size $k$, or
- Green complete graph of size $l$

**Theorem:** (Erdős–Szekeres 1935)

$$R(k, l) \leq \binom{k + l - 2}{k - 1}$$

In particular

$$R(k, k) \leq \binom{2k - 2}{k - 1} \approx 2^{2k}$$
**Proof: part I**

**Induction** on $k+l$. By definition, $R(2,l) = l$ and $R(k,2) = k$. Now suppose that

$$R(a,b) \leq \binom{a+b-2}{a-1}, \quad \forall \ a+b < k+l.$$ 

Let

$$N = R(k-1,l) + R(k,l-1)$$

and consider a red-green coloring of the edges of the complete graph $K_N$.

Fix some vertex $v$ of $K_N$ and let $A$, $B$ be the set of vertices connected to $v$ by red, green edges respectively. Since $|A| + |B| = N - 1$ we have that

$$|A| \geq R(k-1,l) \quad \text{or} \quad |B| \geq R(k,l-1).$$
Proof: part II

If $|A| \geq R(k - 1, l)$, then $A$ must contain either a green clique of size $l$ or a red clique of size $k - 1$ that together with $v$ gives red clique of size $k$ and we are done. The case $|B| \geq R(k, l - 1)$ is similar.

By induction hypothesis, this implies

$$R(k, l) \leq N = R(k - 1, l) + R(k, l - 1)$$

$$\leq \binom{k + l - 3}{k - 2} + \binom{k + l - 3}{k - 1}$$

$$= \binom{k + l - 2}{k - 1}.$$
Growth rate of $R(k, k)$

Example:

$k - 1$ parts

of

size $k - 1$

Conjecture: (P. Turán)

$R(k, k)$ has polynomial growth in $k$,

moreover

\[ R(k, k) \leq c k^2 \]
Erdős existence argument

**Theorem:** (Erdős 1947)

\[ R(k, k) \geq 2^{k/2} \]

**Proof:**
Color the edges of the complete graph \( K_N \) with \( N = 2^{k/2} \) red and green randomly and independently with probability 1/2. For any set \( C \) of \( k \) vertices the probability that \( C \) spans a monochromatic clique is \( 2 \cdot 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}} \).

Since there are \( \binom{N}{k} \) possible choices for \( C \), the probability that coloring contains a monochromatic \( k \)-clique is at most

\[
\binom{N}{k} 2^{1-\binom{k}{2}} \leq \frac{N^k}{k!} \cdot \frac{2^{k/2+1}}{2^{k^2/2}} = \frac{2^{k/2+1}}{k!} \ll 1
\]
Open problem

Determine the correct exponent in the bound for $R(k, k)$

Best current estimates

$$\frac{k}{2} \leq \log_2 R(k, k) \leq 2k$$
Large girth and large chromatic number

Definitions:

- The girth $g(G)$ of a graph is the length of the shortest cycle in $G$.
- The chromatic number $\chi(G)$ is the minimal number of colors which needed to color the vertices of $G$ so that adjacent vertices get different colors.

Note:

It is easy to color graph with large girth "locally" using only three colors.

Question:

If girth of $G$ is large, can it be colored by few colors?
Theorem: (P. Erdős 1959.) For all $k$ and $l$ there exists a finite graph $G$ with girth at least $l$ and chromatic number at least $k$.

Remark:

Explicit constructions of such graphs were not found until only nine years later in 1968 by Lovász.
Bound on $\chi(G)$

**Definition:**
A set of pairwise nonadjacent vertices of a graph $G$ is called independent. The independence number $\alpha(G)$ is the size of the largest independent set in $G$.

**Lemma:**
For every graph $G$ on $n$ vertices

$$\chi(G) \geq \frac{n}{\alpha(G)}.$$

**Proof:**
Consider the coloring of $G$ into $\chi(G)$ colors. Then one of the colors classes has size at least $n/\chi(G)$ and its vertices form an independent set. Thus $\alpha(G) \geq n/\chi(G)$, as desired.
**Probabilistic tools**

*Lemma:* (Linearity of expectation.)

Let $X_1, X_2, \ldots, X_n$ be random variables. Then

$$E \left[ \sum_i X_i \right] = \sum_i E[X_i].$$

*(No conditions on random variables!)*

*Lemma:* (Markov’s inequality.)

Let $X$ be a non-negative random variable and $\lambda$ a real number. Then

$$P[X \geq \lambda] \leq \frac{E[X]}{\lambda}.$$
Fix $\theta < 1/l$. Let $n$ be sufficiently large and $G$ be a random graph $G(n, p)$ with $p = 1/n^{1-\theta}$. Let $X$ be the number of cycles in $G$ of length at most $l$.

As $\theta \cdot l < 1$, by linearity of expectation,

$$
E[X] \leq \sum_{i=3}^{l} n^i \cdot p^i \leq O(n^{\theta l}) = o(n).
$$

By Markov's inequality

$$
P[X \geq n/2] \leq \frac{E[X]}{n/2} = o(1).
$$

Set $x = \frac{3}{p} \log n$, so that

$$
P[\alpha(G) \geq x] \leq \begin{pmatrix} n \\ x \end{pmatrix} (1 - p)^{\binom{x}{2}}
< \left(n e^{-px/2}\right)^x = o(1)
$$
Proof: part II

For large $n$ both of these events have probability less than $1/2$. Thus there is a specific graph $G$ with less than $n/2$ short cycles, i.e., cycles of length at most $l$, and with

$$\alpha(G) < x \leq 3n^{1-\theta} \log n.$$  

Remove a vertex from each short cycle of $G$. This gives $G'$ with at least $n/2$ vertices, girth greater than $l$ and $\alpha(G') \leq \alpha(G)$. Therefore\[\chi(G') \geq \frac{|G'|}{\alpha(G')} \geq \frac{n/2}{3n^{1-\theta} \log n} = \frac{n^\theta}{6 \log n} \gg k.\]
**Set-pair estimate**

**Theorem:** (Bollobás 1965.)

Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be two families of sets such that $A_i \cap B_j = \emptyset$ only if $i = j$. Then

$$m \sum_{i=1}^{m} \left( \frac{|A_i| + |B_i|}{|A_i|} \right)^{-1} \leq 1.$$ 

In particular if $|A_i| = a$ and $|B_i| = b$, then

$$m \leq \binom{a+b}{a}.$$ 

**Example:**

Let $X$ be a set of size $a+b$ and consider pairs $(A_i, B_i = X - A_i)$ for all $A_i \subset X$ of size $a$. There are $\binom{a+b}{a}$ such pairs, so the above theorem is tight.
Proof: part I

Let \(|A_i| = a_i, |B_i| = b_i\) and let

\[ X = \bigcup_i (A_i \cup B_i). \]

Consider a random order \(\pi\) of \(X\) and let \(X_i\) be the event that in this order all the elements of \(A_i\) precede all those of \(B_i\).

To compute probability of \(X_i\) note that there are \((a_i + b_i)!\) possible orders of element in \(A_i \cup B_i\) and the number of such orders in which all the elements of \(A_i\) precede all those of \(B_i\) is exactly \(a_i!b_i!\). Therefore

\[ \mathbb{P}[X_i] = \frac{a_i!b_i!}{(a_i + b_i)!} = \binom{a_i + b_i}{a_i}^{-1}. \]
We claim that events \( X_i \) are pairwise disjoint. Indeed suppose that there is an order of \( X \) in which all the elements of \( A_i \) precede those of \( B_i \) and all the elements of \( A_j \) precede all those of \( B_j \). W.l.o.g. assume that the last element of \( A_i \) appear before the last element of \( A_j \). Then all the elements of \( A_i \) precede all those of \( B_j \), contradicting the fact that \( A_i \cap B_j \neq \emptyset \).

Therefore events \( X_i \) are pairwise disjoint and so we get

\[
1 \geq \sum_{i=1}^{m} \mathbb{P}[X_i] = \sum_{i=1}^{m} \left( \frac{a_i + b_i}{a_i} \right)^{-1}.
\]
Sperner’s lemma

**Theorem:** (Sperner 1928.)

Let $A_1, \ldots, A_m$ be a family of subsets of $n$ element set $X$ which is an antichain, i.e., $A_i \nsubseteq A_j$ for all $i \neq j$. Then

$$m \leq \binom{n}{\lfloor n/2 \rfloor}.$$

**Proof:**

Let $B_i = X - A_i$ and let $|A_i| = a_i$. Then $|B_i| = b_i = n - a_i$, $A_i \cap B_i$ is empty but $A_j \cap B_i \neq \emptyset$ for all $i \neq j$. Therefore by Bollobás’ theorem

$$1 \geq \sum_{i=1}^{m} \left( \frac{a_i + b_i}{a_i} \right)^{-1} = \sum_{i=1}^{m} \left( \frac{n}{a_i} \right)^{-1} \geq \frac{m}{\binom{n}{\lfloor n/2 \rfloor}}.$$
Theorem: (Erdős 1945.)

Let $x_1, x_2, \ldots, x_n$ be real numbers such that all $|x_i| \geq 1$. For every sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \{-1, +1\}$ let

$$x_\alpha = \sum_{i=1}^{n} \alpha_i x_i.$$ 

Then every open interval $I$ in the real line of length 2 contains at most $\left(\lfloor n/2 \rfloor \right)$ of the numbers $x_\alpha$.

Remark:

Kleitman (1970) proved that this is still true if $x_i$ are vectors in arbitrary normed space.
Proof

Replacing $x_i < 0$ by $-x_i$ we can assume that all $x_i \geq 1$. For every $\alpha \in \{-1, 1\}^n$ let $A_\alpha$ be the subset of $\{1, \ldots, n\}$ containing all $1 \leq i \leq n$ with $\alpha_i = -1$. Note that if $A_\alpha \subset A_\beta$ then $\alpha_i - \beta_i$ is either 0 or 2. Hence

$$x_\alpha - x_\beta = \sum_i (\alpha_i - \beta_i)x_i = 2 \sum_{i \in A_\beta - A_\alpha} x_i \geq 2.$$  

This implies that $\{A_\alpha \mid x_\alpha \in I\}$ form an antichain and by Sperner’s lemma their number is bounded by $\binom{n}{\lfloor n/2 \rfloor}$. 
Explicit constructions

**Theorem:** (Erdős 1947)

There is a 2-edge-coloring of complete graph \( K_N, N = 2^{k/2} \) with no monochromatic clique of size \( k \).

**Problem:** (Erdős $100)

Find an "explicit" such coloring.

Explicit \( \overset{\text{def}}{=} \) constructible in polynomial time

**Theorem:** (Frankl and Wilson 1981)

There is an explicit 2-edge-coloring of complete graph \( K_N, N = k^{c \frac{\log k}{\log \log k}} \) with no monochromatic clique of size \( k \).
Bipartite Ramsey

Problem:

Find "large" $0,1$ matrix $A$ with no $k \times k$ homogeneous submatrices.

Submatrix $\overset{\text{def}}{=} \text{intersection of } k \text{ rows and columns}$

Homogeneous $\overset{\text{def}}{=} \text{containing all } 0 \text{ or all } 1$

Randomly:

There is $N \times N$ matrix $A$ with $k = 2 \log_2 N$.

Explicitly:

There is $N \times N$ matrix $A$ with $k = N^{1/2}$.

E.g., take $[N] = \{0,1\}^n$ and define

$$a_{x,y} = x \cdot y \pmod{2}$$
Breaking 1/2 barrier

**Theorem:** (Barak, Kindler, Shaltiel, S., Wigderson)

For every constant $\delta > 0$ there exists a polynomial time computable $N \times N$ matrix $A$ with 0, 1 entries such that none of its $N^\delta \times N^\delta$ submatrices is homogeneous.

Moreover, every $N^\delta \times N^\delta$ submatrix of $A$ has constant proportion of 0 and of 1.