On the function of Erdős and Rogers

Andrzej Dudek*

Department of Mathematical Sciences
Carnegie Mellon University, Pittsburgh, PA 15213, USA
adudek@andrew.cmu.edu

Vojtěch Rödl†

Department of Mathematics and Computer Science
Emory University, Atlanta, GA 30322, USA
rodl@mathcs.emory.edu

October 21, 2009

Abstract

The well-known Ramsey number $R(t,u)$ is the smallest integer n such that every K_t-free graph of order n contains a subset of u vertices with no K_2. Erdős and Rogers considered a more general problem replacing K_2 by K_s for $2 \leq s < t$. Extending the problem of determining Ramsey numbers they defined the following function $f_{s,t}(n) = \min\big\{ \max\{|S| : S \subseteq V(H) \text{ and } H[S] \text{ contains no } K_s\}\big\}$, where the minimum is taken over all K_t-free graphs H of order n. In this paper, we present some old and recent developments concerning function $f_{s,t}(n)$.

1 Introduction

In 1930, Frank Ramsey published a seminal paper "On a problem of formal logic" [13] beginning a new area of research known today as Ramsey theory (for a comprehensive introduction to Ramsey theory see, *e.g.*, [9]). In particular, Ramsey proved that for given integers t and u there is an integer n such that any blue-red coloring of the edges of the complete graph K_n yields either a blue copy of K_t or a red copy of K_u. Such smallest integer n is denoted by $R(t,u)$. In other words, $R(t,u)$ is the smallest integer n such that every K_t-free graph of order n contains an independent set of

*Corresponding author
†Research partially supported by NSF grant DMS 0800070

size u, or equivalently, it contains a u-subset of vertices with no K_2. One can consider a more general problem replacing K_2 by K_s for some $2 \le s < t$. Following this approach in 1962 Erdős and Rogers [7] introduced the following function. For fixed integers $2 \le s < t$ let

$$f_{s,t}(n) = \min \Big\{ \max\{|S| : S \subseteq V(H) \text{ and } H[S] \text{ contains no } K_s\} \Big\},$$

where the minimum is taken over all K_t-free graphs H of order n. (As a matter of fact a variation of this function was already considered by Erdős and Gallai [6].) We will comment first on the meaning of $l \le f_{s,t}(n) < u$. To prove the lower bound it means to show that every K_t-free graph of order n contains a subset of l vertices with no copy of K_s. To prove the upper bound it requires to construct a K_t-free graph of order n such that every subset of u vertices contains a copy of K_s.

As we have just seen the problem of determining $f_{s,t}(n)$ extends that of determining Ramsey numbers. More precisely,

$$R(t,u) = \min\{n : f_{2,t}(n) \ge u\}.$$

Therefore, the problem of determining the precise value of $f_{s,t}(n)$ for $2 \le s < t$ is rather hopeless. Beside of that, this function has attracted a considerable amount of attention and has been studied by several researchers.

In this paper, we review both old and recent results on function $f_{s,t}(n)$.

2 The most restrictive case

In this section we consider the case when $t = s+1$. Erdős and Rogers [7] proved using a probabilistic argument that

$$f_{s,s+1}(n) \le O\Big(n^{1-1/O(s^4 \log s)}\Big). \tag{1}$$

Their proof is based on the concentration of measure phenomenon in the high-dimensional sphere. A similar idea was also used by Alon and Krivelevich [1] who gave an elegant construction of a K_{s+1}-free graph such that every subset of $O\big(n^{1-1/O(s^4 \log s)}\big)$ vertices contains a copy of K_s (see Section 2.1 for details).

Bollobás and Hind [2] improved (1) showing that for any $\varepsilon > 0$ and $s \ge 3$,

$$f_{s,s+1}(n) \le O\Big(n^{1-\frac{s+3}{(s-2)(s+1)}+\varepsilon}\Big). \tag{2}$$

Moreover, they gave the first lower bound showing that for $s \ge 2$,

$$\Omega\big(n^{\frac{1}{2}}\big) \le f_{s,s+1}(n). \tag{3}$$

Subsequently, Krivelevich [10, 11] improved the previous bounds and showed that

$$\Omega\Big(n^{\frac{1}{2}}(\log\log n)^{\frac{1}{2}}\Big) \le f_{s,s+1}(n) \le O\Big(n^{1-\frac{2}{s+2}}(\log n)^{\frac{1}{s-1}}\Big). \tag{4}$$

Summarizing the previous results one can see that the best bounds have essentially been of the form

$$\Omega\big(n^{\frac{1}{2}+o(1)}\big) \leq f_{s,s+1}(n) \leq O\big(n^{1-\epsilon(s)}\big),$$

where $\epsilon(s)$ tends to zero as s goes to infinity. This raised the following question asked by Krivelevich [10] and later by Sudakov [14, 15]. Is it true that for every $0 < \delta < 1$ and s sufficiently large $f_{s,s+1}(n)$ is greater than $n^{1-\delta}$? We showed in [4] that this is not the case proving that for every fixed integer $s \geq 2$,

$$f_{s,s+1}(n) \leq O\big(n^{\frac{2}{3}}\big). \tag{5}$$

In the next sections we present proofs of (1), (3), and (5).

2.1 Proof of $f_{s,s+1}(n) \leq O\big(n^{1-1/O(s^4 \log s)}\big)$ (Alon and Krivelevich [1])

We start with notation. For two vectors $\bar{x} = (x_1, \ldots, x_k)$ and $\bar{y} = (y_1, \ldots, y_k)$ of integers we define the *Hamming distance* between $\bar{x}$ and $\bar{y}$ as,

$$d(\bar{x}, \bar{y}) = \big|\{i : x_i \neq y_i \text{ and } 1 \leq i \leq k\}\big|.$$

Moreover, for a nonempty subset $U \subseteq [s]^k$ and a vector $\bar{x} \in [s]^k$ denote the distance between $\bar{x}$ and U as,

$$d(\bar{x}, U) = \min\{d(\bar{x}, \bar{y}) : \bar{y} \in U\}.$$

Finally, for a given $\delta > 0$ define the δ-*neighborhood* $U_{(\delta)}$ of a nonempty subset $U \subseteq [s]^k$ as,

$$U_{(\delta)} = \{\bar{x} \in [s]^k : d(\bar{x}, U) \leq \delta\}.$$

Now we construct a graph $H = (V, E)$ as follows. Let $s \geq 2$ be a fixed integer and let k be a sufficiently large positive integer (in particular $k \geq s$). Let

$$V = [s]^k \quad \text{and} \quad E = \left\{\{\bar{x}, \bar{y}\} \in V^2 : d(\bar{x}, \bar{y}) > k\left(1 - 1/\tbinom{s+1}{2}\right)\right\}.$$

First we show that H is a K_{s+1}-free graph. Assume by contradiction that there are $\bar{x}^1, \ldots, \bar{x}^{s+1} \in V$ such that $H[\{\bar{x}^1, \ldots, \bar{x}^{s+1}\}] = K_{s+1}$, *i.e.*,

$$d(\bar{x}^i, \bar{x}^j) > k\left(1 - 1/\tbinom{s+1}{2}\right) \tag{6}$$

for every $1 \leq i \neq j \leq s+1$. Since every coordinate can attain only s distinct values we infer that for every coordinate h, $1 \leq h \leq k$, there is a pair of vertices in K_{s+1} having the same value of the h-th coordinate. Thus,

$$\sum_{1 \leq i < j \leq s+1} d(\bar{x}^i, \bar{x}^j) \leq k\binom{s+1}{2} - k = \binom{s+1}{2}\left(k - k/\binom{s+1}{2}\right),$$

and consequently, there is a pair $\bar{x}^i$, $\bar{x}^j$ of vertices of K_{s+1} such that $d(\bar{x}^i, \bar{x}^j) \leq k - k/\binom{s+1}{2}$ which contradicts (6).

It remains to show that every sufficiently large subset of vertices of H contains a copy of K_s. First we show that every sufficiently large subset of vertices of H contains an s-simplex defined as follows. An *s-simplex* is an s-subset $\{\bar{x}^1, \ldots, \bar{x}^s\}$ of vertices V so that

$$d(\bar{x}^i, \bar{x}^j) = k \text{ for every } 1 \leq i \neq j \leq s. \tag{7}$$

For simplicity we also endow V with the normalized counting measure P (which can be also viewed as the probability measure) defined as $P(A) = \frac{|A|}{|V|}$ for every $A \subseteq V$.

Proposition 2.1 *If $V_0 \subseteq V$ and $P(V_0) > \frac{s-1}{s}$, then V_0 contains an s-simplex.*

Proof. By symmetry, every vertex of V belongs to the same number of s-simplices. Clearly, if an s-simplex S is not in V_0, then at least one vertex of S is not in V_0. Hence, choosing randomly and uniformly a simplex S among all s-simplices in V we infer that

$$P(S \nsubseteq V_0) \leq s\frac{|V| - |V_0|}{|V|} = s(1 - P(V_0)) < s\left(1 - \frac{s-1}{s}\right) = 1,$$

and thus, there is an s-simplex S in V_0. □

In order to finish the proof we will need the following isoperimetric inequality. For the proof see, *e.g.*, Lemma 3.3 in [1] or Proposition 7.12 in [12].

Proposition 2.2 *For $c > 0$ define $\delta(c) = \lceil\sqrt{k}(\sqrt{(\log s)/2} + c)\rceil$. If $U \subseteq V$ and $P(U) \geq \exp(-2c^2)$, then $P(U_{(\delta)}) > \frac{s-1}{s}$.*

Let

$$c = \frac{\sqrt{k}}{2\binom{s+1}{2}} - \sqrt{(\log s)/2} - 1 = \frac{\sqrt{k}}{2\binom{s+1}{2}}(1 + o(1)), \tag{8}$$

where $o(1)$ tends to zero as k tends to infinity. Note that

$$\delta(c) = \left\lceil\sqrt{k}(\sqrt{(\log s)/2} + c)\right\rceil = \left\lceil k/2\tbinom{s+1}{2} - \sqrt{k}\right\rceil < \frac{k}{2\binom{s+1}{2}} - \sqrt{k} + 1. \tag{9}$$

Recall that $|V| = s^k$ and so by (8)

$$\exp(-2c^2) = |V|^{-2c^2 \log_{|V|} \mathrm{e}} = |V|^{-\frac{2c^2}{\log|V|}} = |V|^{-\frac{2c^2}{k \log s}} = |V|^{-\frac{(2+o(1))}{s^2(s+1)^2 \log s}}.$$

We show that every subset $U \subseteq V$ of size

$$|U| = |V| \exp(-2c^2) = |V|^{1-\frac{(2+o(1))}{s^2(s+1)^2 \log s}}$$

contains a copy of K_s. Indeed, if $P(U) \geq \exp(-2c^2)$, then by Proposition 2.2 $P(U_{(\delta)}) > \frac{s-1}{s}$, where $\delta = \lceil \sqrt{k}(\sqrt{(\log s)/2} + c) \rceil$. Hence, by Proposition 2.1 (applied to $V_0 = U_{(\delta)}$) the set $U_{(\delta)}$ contains an s-simplex S (*cf.* (7)). Let $\bar{x}^1, \ldots, \bar{x}^s \in U_{(\delta)}$ be its vertices. It remains to show that the set U induces a copy of K_s.

It follows from the definition of $U_{(\delta)}$ that for every $\bar{x}^i$ there is $\bar{y}^i \in U$ such that

$$d(\bar{x}^i, \bar{y}^i) \leq \delta, \tag{10}$$

where $i = 1, \ldots, s$. Let $1 \leq i \neq j \leq s$. Then, by the triangle inequality

$$d(\bar{x}^i, \bar{x}^j) \leq d(\bar{x}^i, \bar{y}^i) + d(\bar{y}^i, \bar{x}^j) \leq d(\bar{x}^i, \bar{y}^i) + d(\bar{y}^i, \bar{y}^j) + d(\bar{y}^j, \bar{x}^j),$$

and consequently by (10),

$$d(\bar{x}^i, \bar{x}^j) \leq d(\bar{y}^i, \bar{y}^j) + 2\delta.$$

Hence, since $d(\bar{x}^i, \bar{x}^j) = k$ and by (9) $2\delta < k/\binom{s+1}{2} - 2\sqrt{k} + 2 < k/\binom{s+1}{2}$,

$$d(\bar{y}^i, \bar{y}^j) \geq d(\bar{x}^i, \bar{x}^j) - 2\delta > k - k/\binom{s+1}{2}.$$

That means that the vertices $\bar{y}^1, \ldots, \bar{y}^s \in U$ form a copy of K_s in H.

This completes the proof of (1).

2.2 Proof of $\Omega\big(n^{\frac{1}{2}}\big) \leq f_{s,s+1}(n)$ for $s \geq 2$ (Bollobás and Hind [2])

We show that for any $s \geq 2$ and n large enough

$$\lfloor ((s-1)n)^{\frac{1}{2}} \rfloor \leq f_{s,s+1}(n). \tag{11}$$

Let G be a K_{s+1}-free graph of order n. We are going to show that G contains a set of $\lfloor ((s-1)n)^{\frac{1}{2}} \rfloor$ vertices with no copy of K_s. Let v be a vertex of maximal degree and let W be the set of neighbors of v. Clearly, $G[W]$ is K_s-free, and hence, if $|W| \geq \lfloor ((s-1)n)^{\frac{1}{2}} \rfloor$ then we are done. Therefore, we may assume that $|W| < \lfloor ((s-1)n)^{\frac{1}{2}} \rfloor$. Consequently, the chromatic number of G satisfies

$$\chi(G) \leq |W| + 1 \leq \lfloor ((s-1)n)^{\frac{1}{2}} \rfloor. \tag{12}$$

Let $W_1, \ldots, W_{s-1}$ be color classes in a $\chi(G)$-vertex-coloring of G such that $|W_1 \cup \cdots \cup W_{s-1}|$ is maximal. Clearly $G[W_1 \cup \cdots \cup W_{s-1}]$ is K_s-free. Moreover, by (12),

$$|W_1 \cup \cdots \cup W_{s-1}| \geq (s-1)\frac{n}{\chi(G)} \geq \lfloor ((s-1)n)^{\frac{1}{2}} \rfloor,$$

and hence, (11) holds, as required.

2.3 Proof of $f_{s,s+1}(n) \leq O\big(n^{\frac{2}{3}}\big)$ for $s \geq 2$ [4]

The main idea we employ here is similar to an approach taken in [3].

First we recall some basic properties of generalized quadrangles. A *generalized quadrangle* $Q(4,q)$ is an incidence structure on a set P of points and a set $\mathcal{L}$ of lines such that:

(Q1) any two points lie in at most one line,

(Q2) if u is a point not on a line L, then there is a unique point $w \in L$ collinear with u, and hence, no three lines form a triangle,

(Q3) every line contains $q+1$ points, and every point lies on $q+1$ lines.

It is known that for every prime power q such an incidence structure $Q(4,q)$ exists with $|P| = |\mathcal{L}| = q^3+q^2+q+1$. For more information about generalized quadrangles see [8, 16].

Fix an integer $s \geq 2$. For every m we are going to construct a K_{s+1}-free graph H of order $\Theta(m^3)$ such that any induced subgraph of order $\lfloor |V(H)|/m \rfloor$ contains a copy of K_s. Thus, setting $n = (cm)^3$, $c = c(s)$, implies

$$\lfloor |V(H)|/m \rfloor \leq c^3 m^2 = cn^{2/3},$$

and consequently, (5) holds.

By Bertrand's postulate there is a prime number q such that

$$s^2 m \leq q+1 \leq 2s^2 m.$$

Let $Q(4,q)$ be a generalized quadrangle with a set P of points and a set $\mathcal{L}$ of lines. We construct a "random graph" H with vertex set P. Clearly,

$$|V(H)| = |P| = q^3+q^2+q+1 = \Theta(m^3).$$

First, we partition every line $L \in \mathcal{L}$ into s sets of the same size ℓ (for simplicity, we assume that s divides $q+1$), hence

$$\ell = \frac{q+1}{s} \geq sm. \tag{13}$$

More precisely, for each line L we choose one ordered partition

$$L = \bigcup_{i=1}^{s} L_i$$

satisfying $|L_1| = \cdots = |L_s| = \ell$ randomly and uniformly from the set of all such partitions. Now we join every $u \in L_i$ and $w \in L_j$, $1 \leq i < j \leq s$, by an edge obtaining a complete s-partite graph of order $s\ell$. Note that by (Q1) every edge is determined by a unique line. Moreover, condition (Q2) yields that H contains no clique of size $s+1$. We show that in some such graph H (randomly chosen from the space of such graphs) every set $U \subseteq V(H)$, $|U| = \lfloor |P|/m \rfloor = \Theta(m^2)$, contains a copy of K_s.

For $U \subseteq V(H)$ with cardinality $|U| = \lfloor |P|/m \rfloor$ let $\mathcal{A}(U)$ be the event that K_s is not a subgraph of $H[U]$. Clearly, $\mathcal{A}(U)$ implies $\mathcal{A}(L \cap U)$ for each $L \in \mathcal{L}$. Consequently,

$$\mathcal{A}(U) \subseteq \bigcap_{L \in \mathcal{L}} \mathcal{A}(L \cap U),$$

and since all events $\mathcal{A}(L \cap U)$ are independent,

$$\Pr(\mathcal{A}(U)) \leq \prod_{L \in \mathcal{L}} \Pr(\mathcal{A}(L \cap U)). \tag{14}$$

For a fixed line $L \in \mathcal{L}$ we bound from above the probability that $\mathcal{A}(L \cap U)$ occurs. Let $L = \bigcup_{i=1}^{s} L_i$ be a partition of L. Note that if $\mathcal{A}(L \cap U)$ happens then for some i, $1 \leq i \leq s$, $L_i \cap U = \emptyset$, *i.e.*, L_i and U are disjoint. Let $|U \cap L| = u_L$. The probability that for a fixed i, $1 \leq i \leq s$, $U \cap L_i = \emptyset$ equals the probability that for a fixed partition $L = \bigcup_{i=1}^{s} L_i$ a randomly chosen subset T with $|T| = u_L$ satisfies $T \cap L_i = \emptyset$. Hence,

$$\Pr(\mathcal{A}(L \cap U)) \leq s \frac{\binom{q+1-\ell}{u_L}}{\binom{q+1}{u_L}} \leq s \left(\frac{q+1-\ell}{q+1} \right)^{u_L} \leq s \exp\left(-\frac{\ell u_L}{q+1} \right).$$

Consequently, (14) yields,

$$\Pr(\mathcal{A}(U)) \leq s^{|\mathcal{L}|} \exp\left(-\frac{\ell}{q+1} \sum_{L \in \mathcal{L}} u_L \right).$$

Moreover, since every point in U belongs to exactly $q+1$ lines

$$\sum_{L \in \mathcal{L}} u_L = \sum_{L \in \mathcal{L}} |U \cap L| = |U|(q+1).$$

Hence,

$$\Pr(\mathcal{A}(U)) \leq s^{|\mathcal{L}|} \exp\left(-\ell |U|\right) = s^{|P|} \exp\left(-\ell |U|\right).$$

This implies that

$$\Pr\left(\bigcup_{U} \mathcal{A}(U) \right) \leq \binom{|P|}{\lfloor |P|/m \rfloor} s^{|P|} \exp\left(-\ell|U|\right) \leq (\mathrm{e}m)^{|P|/m} s^{|P|} \exp\left(-\ell |U|\right),$$

where the union is taken over all subsets $U \subseteq V(H)$ with cardinality $\lfloor |P|/m \rfloor$. Finally, note that by (13) $\ell|U| \geq sm\lfloor |P|/m \rfloor \geq \frac{99}{100}s|P|$, and hence,

$$\Pr\left(\bigcup_U \mathcal{A}(U)\right) \leq \exp\left(|P|\left(\frac{1}{m} + \frac{\log m}{m} + \log s - \frac{99}{100}s\right)\right),$$

which tends to zero as m tends to infinity.

This completes the proof of (5).

3 General bounds

Here we discuss general bounds on $f_{s,t}(n)$ for any $t \geq s+1$. First observe that for any $t \geq s+1$,

$$f_{s,t}(n) \leq f_{s,s+1}(n).$$

Consequently, (1), (2), (4), and (5) trivially bound from above $f_{s,t}(n)$. In particular, for every $t \geq s+1$,

$$f_{s,t}(n) \leq O\big(n^{\frac{2}{3}}\big).$$

Moreover, some of the previous results can be easily generalized. For instance, Bollobás and Hind [2] showed

$$\Omega\big(n^{\frac{1}{t-s+1}}\big) \leq f_{s,t}(n).$$

Subsequently, Krivelevich [10, 11] slightly improved this lower bound and also gave a new general upper bound,

$$\Omega\big(n^{\frac{1}{t-s+1}}(\log\log n)^{1-\frac{1}{t-s+1}}\big) \leq f_{s,t}(n) \leq O\big(n^{\frac{s}{t+1}}(\log n)^{\frac{1}{s-1}}\big). \tag{15}$$

Recently, Sudakov [14, 15] improved the lower bound showing that

$$\Omega\big(n^{\frac{s}{2t}+O(1/t^2)}\big) \leq f_{s,t}(n).$$

In fact, he showed a more general result. For a fixed $s \geq 3$ consider a sequence $\{a_i\}_{i=3-s}^{\infty}$ defined as $a_i = 1$ for every $3-s \leq i \leq 0$, $a_1 = \frac{3s-4}{5s-6}$, and $\frac{1}{a_i} = 1 + \frac{1}{s-1}\sum_{j=i-(s-1)}^{i-1}\frac{1}{a_j}$ for every $i \geq 2$. Then, for any $s \geq 3$ and $k \geq 2$,

$$\Omega\left(n^{a_k(s)}\right) \leq f_{s,s+k}(n). \tag{16}$$

We found in [3] an explicit formula for $a_k(s)$ in a special case when $s \geq k \geq 2$ showing that $\frac{1}{a_k(s)} = 1 + \frac{3s-2}{3s-4}\left(\frac{s}{s-1}\right)^{k-2}$. Consequently, for any $s \geq k \geq 2$,

$$\Omega\Big(n^{1/\left(1+\frac{3s-2}{3s-4}(\frac{s}{s-1})^{k-2}\right)}\Big) \leq f_{s,s+k}(n).$$

Furthermore, we showed [3] that for any $\varepsilon > 0$, $k \geq 1$, and sufficiently large $s \geq s_0 = s_0(\varepsilon, k)$,

$$f_{s,s+k}(n) \leq O\Big(n^{\frac{k+1}{2k+1}+\varepsilon}\Big). \tag{17}$$

Consequently, for every $\varepsilon > 0$ and sufficiently large integers $k \geq k_0 = k_0(\varepsilon)$ and $s \geq s_0 = s_0(\varepsilon, k)$,

$$\Omega\Big(n^{\frac{1}{2}-\varepsilon}\Big) \leq f_{s,s+k}(n) \leq O\Big(n^{\frac{1}{2}+\varepsilon}\Big). \tag{18}$$

Below we present the main ideas from the proofs of (16) and (17).

3.1 Proof of $\Omega\big(n^{a_k(s)}\big) \leq f_{s,s+k}(n)$ (Sudakov [14, 15])

For simplicity we show a slightly weaker result which was also proved by Sudakov [14]. We define a new sequence of integers $\{b_i\}$ which is easier to handle. Let $s \geq 4$ be a fixed integer and $\{b_i\}_{i=2-s}^{\infty}$ be defined as $b_i = 1$ for every $2 - s \leq i \leq 0$ and $\frac{1}{b_i} = 1 + \frac{1}{s-1} \sum_{j=i-(s-1)}^{i-1} \frac{1}{b_j}$ for every $i \geq 1$. It was shown in [14] that for any $k \geq 0$,

$$\Omega\left(n^{b_k}\right) \leq f_{s,s+k}(n). \tag{19}$$

Since for all $i \geq 1$, $b_i < a_i$, (19) is weaker than (16).

In order to prove (19) we need two auxiliary results which we state without proofs (for details see Lemma 3.1 and 3.2 in [15]). The first one is the well-known estimate on the size of maximum independent size in s-uniform hypergraph.

Proposition 3.1 *Let $\mathcal{H}$ be an s-uniform hypergraph on n vertices with $m \geq \Omega(n)$ edges. Then, $\mathcal{H}$ contains an independent set of size*

$$\alpha(\mathcal{H}) \geq \Omega\left(\frac{n^{\frac{s}{s-1}}}{m^{\frac{1}{s-1}}}\right).$$

The next proposition gives an estimate on the number of edges in an s-uniform hypergraph. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and $U \subseteq V$. We denote by $N_{\mathcal{H}}(U)$ the neighborhood of U in $\mathcal{H}$, *i.e.*,

$$N_{\mathcal{H}}(U) = \Big\{\bigcup E : U \subseteq E \text{ and } E \in \mathcal{E}\Big\} \setminus U.$$

Proposition 3.2 *Let $\mathcal{H} = (V, \mathcal{E})$ be an s-uniform hypergraph of order n. Then, the number of edges is bounded by*

$$|\mathcal{E}| \leq O\left(n \prod_{u=1}^{s-1} \max_{U \subseteq V,\ |U|=u} |N_{\mathcal{H}}(U)|\right).$$

Note that if H is a graph (2-uniform hypergraph) of order n and with m edges, then Proposition 3.2 suggests the obvious bound $m \le n\Delta(H)$. Moreover, observe that Proposition 3.2 is tight for complete s-uniform hypergraphs.

The heart of the proof of (19) lies in the following proposition. Denote by $N^*_G(U)$ the set of all common neighbors in $G = (V, E)$ of vertices from $U \subseteq V$, that is,

$$N^*_G(U) = \{v \in V : \{v, u\} \in E \text{ for all } u \in U\} \setminus U.$$

Proposition 3.3 *Let $G = (V, E)$ be a graph of order n such that each of its cliques induced by $U \subseteq V$ of size u, $1 \le u \le s-1$, satisfies*

$$|N^*_G(U)| < n^{\frac{b_{k+1}}{b_{k+1-u}}}. \tag{20}$$

Then G contains a K_s-free subgraph of order $\Omega(n^{b_{k+1}})$.

Proof. Let $\mathcal{H}$ be an s-uniform hypergraph whose vertices are the vertices of G and whose edges are all copies of K_s in G. Clearly, an independent set in $\mathcal{H}$ corresponds to a K_s-free subgraph in G. Therefore, in order to finish the proof it is enough to show that

$$\alpha(\mathcal{H}) \ge \Omega(n^{b_{k+1}}). \tag{21}$$

Also observe that

$$|N_{\mathcal{H}}(U)| \le |N^*_G(U)| \tag{22}$$

for any subset of vertices U of size at most $s-1$ inducing a clique in G. Denote by m the number of edges of $\mathcal{H}$. Then, Proposition 3.2, (22) and (20) yield,

$$\begin{aligned} m &\le O\left(n \prod_{u=1}^{s-1} \max_{U\subseteq V,\ |U|=u} |N_{\mathcal{H}}(U)|\right) \le O\left(n \prod_{u=1}^{s-1} \max_{G[U] \text{ is a clique},\ |U|=u} |N^*_G(U)|\right) \\ &\le O\left(n \prod_{u=1}^{s-1} n^{\frac{b_{k+1}}{b_{k+1-u}}}\right) = O\left(n^{1+\sum_{u=1}^{s-1} \frac{b_{k+1}}{b_{k+1-u}}}\right) = O\left(n^{1+b_{k+1}\sum_{i=0}^{s-2} \frac{1}{b_{k-i}}}\right). \end{aligned} \tag{23}$$

We may also assume that $m \ge \Omega(n)$. For otherwise, if $m \le o(n)$, then $\mathcal{H}$ contains an independent set of size $n - m \ge n - o(n) \ge \Omega(n)$. Thus, Proposition 3.1 and (23) imply that $\mathcal{H}$ contains an independent set of size

$$\alpha(\mathcal{H}) \ge \Omega\left(\frac{n^{\frac{s}{s-1}}}{m^{\frac{1}{s-1}}}\right) \ge \Omega\left(n^{1-\frac{b_{k+1}}{s-1}\sum_{i=0}^{s-2}\frac{1}{b_{k-i}}}\right).$$

Finally note that the recurrence relation for sequence $\{b_i\}$ yields

$$b_{k+1} = 1 - \frac{b_{k+1}}{s-1} \sum_{i=0}^{s-2} \frac{1}{b_{k-i}}.$$

This completes the proof of (21) and so the proof of Proposition 3.3. □

Now we are going to prove (19). For a fixed $s \geq 4$ we show by induction on $k \geq 0$ that any K_{s+k}-free graph G of order n contains a K_s-free subgraph of order $\Omega(n^{b_k})$.

If $k = 0$, then G itself is K_s-free, and hence, $f_{s,s}(n) = n^{b_0}$ for $b_0 = 1$. Next suppose that the statement holds for all $k' \leq k$, *i.e.*, $f_{s,s+k'}(n) \geq \Omega(n^{b_{k'}(s)})$. We show that it also holds for $k+1$. Let G be a $K_{s+(k+1)}$-free graph. We need to show that G contains a subset of $\Omega(n^{b_{k+1}(s)})$ vertices with no copy of K_s.

First let us assume that $k \geq s-2$. Let U be a subset of vertices of G which forms a clique of size $1 \leq u \leq s-1$. Then, $k+1-u \geq 0$ and the subgraph of G induced by $N^*_G(U)$ contains no clique of size $s+(k+1-u)$. Hence, if $|N^*_G(U)| \geq n^{b_{k+1}/b_{k+1-u}}$, then the inductive hypothesis applied to $N^*_G(U)$ yields that $G[N^*_G(U)]$ contains a K_s-free subgraph of size $\Omega(|N^*_G(U)|^{b_{k+1-u}}) \geq \Omega\left(n^{b_{k+1}}\right)$. Thus, we may assume that (20) holds, and consequently, Proposition 3.3 yields (19).

Now let us suppose that $0 \leq k < s-2$. Similarly, as in the previous case one can assume by the inductive hypothesis that $|N^*_G(U)| < n^{b_{k+1}/b_{k+1-u}}$ for every subset U inducing a clique in G, but now only of size $1 \leq u \leq k+1$. We show that this is also true for any $k+1 < u \leq s-1$. Clearly, every clique induced by U of size larger than $k+1$ contains a sub-clique induced by $U' \subseteq U$ of size $k+1$. Since $N^*_G(U) \subseteq N^*_G(U')$ and $b_{-(s-2)} = \cdots = b_0 = 1$, we obtain for $|U| = u$ that nevertheless

$$|N^*_G(U)| \leq |N^*_G(U')| \leq n^{\frac{b_{k+1}}{b_0}} = n^{\frac{b_{k+1}}{b_{k+1-u}}},$$

for all $k+1 < u \leq s-1$. Consequently, we may assume that the assumption of Proposition 3.3 is satisfied and so (19) holds, as required.

3.2 Sketch of the proof of $f_{s,s+k}(n) \leq O\big(n^{\frac{k+1}{2k+1}+\varepsilon}\big)$ for $s \geq s_0 = s_0(\varepsilon, k)$ [4]

First we recall some basic properties of projective planes (for more information see [8, 16]). A *projective plane* $PG(2,q)$ is an incidence structure on a set P of points and a set $\mathcal{L}$ of lines such that:

(P1) any two points lie in a unique line,

(P2) any two lines meet in a unique point,

(P3) every line contains $q+1$ points, and every point lies on $q+1$ lines.

It is known that for every prime power q such an incidence structure $PG(2,q)$ exists with $|P| = |\mathcal{L}| = q^2 + q + 1$.

Fix an arbitrarily small $\varepsilon > 0$ and an integer $k \geq 1$. Let s be a fixed and sufficiently large integer depending on ε and k only. We will show that for every integer m there exists a graph H of order $\lfloor (cm)^{2+\frac{1}{k}+2\varepsilon} \rfloor$, $c = c(s)$, such that H is K_{s+k}-free and any subgraph of H induced by a set of cardinality $\lfloor |V(H)|/m \rfloor$ contains a copy of K_s. Thus, setting $n = (cm)^{2+\frac{1}{k}+2\varepsilon}$ implies

$$\lfloor |V(H)|/m \rfloor \le c^{2+\frac{1}{k}+2\varepsilon} m^{1+\frac{1}{k}+2\varepsilon} = cn^{\frac{k+1+2\varepsilon k}{2k+1+2\varepsilon k}} \le cn^{\frac{k+1}{2k+1}+\varepsilon},$$

and consequently, (17) will hold.

By Bertrand's postulate there is a prime number q such that

$$m^{1+\frac{1}{2k}+\varepsilon} \le q+1 \le 2m^{1+\frac{1}{2k}+\varepsilon}.$$

Let $PG(2,q)$ be a projective plane with a set P of points and a set $\mathcal{L}$ of lines. We construct a "random graph" H with the vertex set P. Clearly,

$$|V(H)| = |P| = q^2+q+1 = \Theta(m^{2+\frac{1}{k}+2\varepsilon}).$$

We partition every line $L \in \mathcal{L}$ into $s+1$ sets

$$L = \bigcup_{i=0}^{s} L_i \tag{24}$$

randomly and uniformly from the set of all partitions (24) satisfying

$$|L_1| = \cdots = |L_s| = \ell = sm \quad \text{and} \quad |L_0| = q+1-s\ell.$$

Observe that $|L_0| \gg \bigcup_{i=1}^{s} |L_i|$ as m tends to infinity. The edges of H are all pairs $\{u,w\}$, where $u \in L_i$, $w \in L_j$ and $1 \le i < j \le s$. In other words, $\bigcup_{i=1}^{s} L_i$ induces a complete s-partite graph of order $s\ell$, which we denote by $K_s(L)$. The edge set of H then equals

$$E(H) = \bigcup_{L\in\mathcal{L}} K_s(L).$$

Note that since every two points from P lie in a unique line the graph H is well-defined.

Denote by $\mathcal{H} = \mathcal{H}(\varepsilon,k,s,q)$ the space of all such graphs H. Note that

$$|\mathcal{H}| = \left(\binom{q+1}{\ell}\binom{q+1-\ell}{\ell}\cdots\binom{q+1-(s-1)\ell}{\ell}\right)^{|\mathcal{L}|} = \left(\frac{(q+1)!}{(\ell!)^s(q+1-s\ell)!}\right)^{|\mathcal{L}|}.$$

One can show that a graph H randomly chosen from the space $\mathcal{H}$ has the following properties:

(i) every set $U \subseteq V(H)$, $|U| = \lfloor |P|/m \rfloor = \Theta(m^{1+\frac{1}{k}+2\varepsilon})$, induces in H a subgraph containing a copy of K_s, and

(ii) H is K_{s+k}-free.

The proof of property (i) is similar to the proof from Section 2.3. The proof of (ii) is more technical and so we also skip it (for details see Theorem 1.2 in [4]).

4 Concluding remarks

In this paper we presented the current stage of research on function $f_{s,t}(n)$. In [5] Erdős asked to estimate $f_{s,t}(n)$ as accurately as possible. In particular, he conjectured the following.

Conjecture 4.1 (Erdős [5]) *For $s+1<t$,*

$$\lim_{n\to\infty} \frac{f_{s+1,t}(n)}{f_{s,t}(n)} = \infty.$$

Sudakov [14] showed that this conjecture holds for

$$(s,t) \in \{(2,4),(2,5),(2,6),(2,7),(2,8),(3,6)\}.$$

Indeed, by (3) we obtain that $f_{3,4}(n) \geq \Omega(n^{\frac{1}{2}})$ and by (16) we get $f_{3,5}(n) \geq \Omega(n^{\frac{5}{12}})$, $f_{3,6}(n) \geq \Omega(n^{\frac{10}{31}})$, $f_{3,7}(n) \geq \Omega(n^{\frac{4}{15}})$, $f_{3,8}(n) \geq \Omega(n^{\frac{40}{177}})$, and $f_{4,6} \geq \Omega(n^{\frac{4}{9}})$. On the other hand, (15) implies $f_{2,4}(n) \leq O(n^{\frac{2}{5}} \log n)$, $f_{2,5}(n) \leq O(n^{\frac{2}{6}} \log n)$, $f_{2,6}(n) \leq O(n^{\frac{2}{7}} \log n)$, $f_{2,7}(n) \leq O(n^{\frac{2}{8}} \log n)$, $f_{2,8}(n) \leq O(n^{\frac{2}{9}} \log n)$, and $f_{3,6}(n) \leq O(n^{\frac{3}{7}} (\log n)^{\frac{1}{2}})$, which imply Sudakov's result.

Motivated by (3), (5) and (18) we state another conjecture.

Conjecture 4.2 *For every $\varepsilon > 0$ and s large enough,*

$$f_{s,s+1}(n) \leq O(n^{\frac{1}{2}+\varepsilon}).$$

In view of (4), if Conjecture 4.2 holds then it is best possible.

5 Acknowledgment

The authors would like to thank Alexander Soifer for organizing the successful DIMACS Workshop "Ramsey Theory Yesterday, Today, and Tomorrow".

References

[1] N. Alon and M. Krivelevich, *Constructive bounds for a Ramsey-type problem*, Graphs Combin. **13** (1997), 217–225.

[2] B. Bollobás and H. R. Hind, *Graphs without large triangle free subgraphs*, Discrete Math. **87** (1991), no. 2, 119–131.

[3] A. Dudek and V. Rödl, *An almost quadratic bound on vertex Folkman numbers*, to appear in J. Combin. Theory Ser. B.

[4] A. Dudek and V. Rödl, *On K_s-free subgraphs in K_{s+k}-free graphs*, submitted.

[5] P. Erdős, *Some of my recent problems in combinatorial number theory, geometry and combinatorics*, in: Graph theory, combinatorics, and algorithms, Vol. 1, 2, Wiley, New York, 1995, 335–349.

[6] P. Erdős and T. Gallai, *On the minimal number of vertices representing the edges of a graph*, Publ. Math. Inst. Hungar. Acad. Sci. **6** (1961), 181–202.

[7] P. Erdős and C. A. Rogers, *The construction of certain graphs*, Canad. J. Math. **14** (1962), 702–707.

[8] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, New York, 2001.

[9] R. Graham, B. Rothschild, and J. Spencer, *Ramsey Theory*, Wiley, New York, 1990.

[10] M. Krivelevich, *Bounding Ramsey numbers through large deviation inequalities*, Random Structures Algorithms **7** (1995), 145–155.

[11] M. Krivelevich, *K^s-free graphs without large K^r-free subgraphs*, Combin. Probab. Comput. **3** (1994), 349–354.

[12] C. J. H. McDiarmid, *On the method of bounded differences*, in *Surveys in Combinatorics* 1989, London Math. Society Lecture Notes Series 141 (Siemons J., ed.), Cambridge Univ. Press (1989), 148–188.

[13] F. Ramsey, *On a problem of formal logic*, Proc. Lond. Math. Soc. **30** (1930), 264–286.

[14] B. Sudakov, *A new lower bound for a Ramsey-type problem*, Combinatorica **25** (2005), 487–498.

[15] B. Sudakov, *Large K_r-free subgraphs in K_s-free graphs and some other Ramsey-type problems*, Random Structures Algorithms **26** (2005), 253-265.

[16] J. Thas, *Generalized polygons*, in: F. Buekenhout (Ed.), Handbook on Incidence Geometry, North-Holland, Amsterdam, 1995, pp. 383–431.