

Cliques in Steiner Systems*

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August 3, 2008

Abstract

A partial Steiner (k, l) -system is a k -uniform hypergraph \mathcal{G} with the property that every l -element subset of V is contained in at most one edge of \mathcal{G} . In this paper we show that for given k, l and t there exists a partial Steiner (k, l) -system such that whenever an l -element

*Dedicated to Professor Alexander Rosa on the occasion of his 70th birthday.

†Research partially supported by NSF grant DMS 0300529

subset from every edge is chosen, the resulting l -uniform hypergraph contains a clique of size t . As the main result of this note, we establish asymptotic lower and upper bounds on the size of such cliques with respect to the order of Steiner systems.

1 Introduction

A *partial Steiner (k,l) -system* ((k,l) -system in short) is a k -uniform hypergraph $\mathcal{G} = (V, \mathcal{E})$ with the property that every l -element subset of V is contained in at most one edge of \mathcal{G} . For fixed k and l we denote the set of all (k,l) -systems by $S(k,l)$. Questions regarding the maximum numbers of edges in (k,l) -systems have been studied, *e.g.*, in [1, 8, 17]. Another direction of the research was pioneered by Alex Rosa [15, 16], who was the first to investigate questions regarding the chromatic number of Steiner systems. This motivated a further study on chromatic numbers and independent sets of Steiner systems by a number of researchers (see, *e.g.*, [4, 5, 6, 9, 12, 18]). The aim of this note is to introduce a Ramsey type parameter related to (k,l) -systems. The following notion of a selector is essential for the discussion. Let $l < k$ be integers and $\mathcal{H} = (V, \mathcal{E})$ be a k -uniform hypergraph. A *selector* is a function $\mathbf{S} : \mathcal{E} \rightarrow [V]^l$ satisfying $\mathbf{S}(E) \subset E$ for every $E \in \mathcal{E}$. Moreover, denote by $K_n^{(l)}$ the l -uniform complete hypergraph of order n . With this concept in mind we start with the following reformulation of the Ramsey theorem (see, *e.g.*, [10, 14]), which in particular says that there exists the smallest integer $R_l(k,t)$ such that any blue-red coloring of the edges of $K_{R_l(k,t)}^{(l)}$ yields either a blue copy of $K_k^{(l)}$ or a red copy of $K_t^{(l)}$.

Theorem 1.1. *Let k, l, t be integers satisfying $l \leq \min\{k, t\}$. Then, there exists n such that the hypergraph $K_n^{(k)} = (V, \mathcal{E})$ has the following property. For any selector $\mathbf{S} : \mathcal{E} \rightarrow [V]^l$ the l -uniform hypergraph $(V, \mathbf{S}(\mathcal{E}))$ contains a clique $K_t^{(l)}$.*

Note that the smallest such integer n equals $R_l(k,t)$. In this note we are interested in an extension of Theorem 1.1 with $(V, \binom{V}{k})$ replaced by a “sparse” hypergraph. Clearly, if (V, \mathcal{E}) is a partial $(k, l-1)$ -system, then for any selector \mathbf{S} and $E, E' \in \mathcal{E}$, $E \neq E'$, $|\mathbf{S}(E) \cap \mathbf{S}(E')| \leq l-2$ holds, and consequently, the l -uniform hypergraph $(V, \mathbf{S}(\mathcal{E}))$ cannot contain a clique $K_t^{(l)}$. We show, however, that with a conveniently chosen (k,l) -system Theorem 1.1 remains true.

Theorem 1.2. *Let k, l, t be integers satisfying $l \leq \min\{k, t\}$. Then, there exists a (k, l) -system $\mathcal{T} = (V, \mathcal{E})$ such that for any selector $S : \mathcal{E} \rightarrow [V]^l$ the l -uniform hypergraph $(V, S(\mathcal{E}))$ contains a clique $K_t^{(l)}$.*

The special case of Theorem 1.2 (for $k = 3$ and $l = 2$) follows from the result of the second author [7], where it was shown that for any positive integer t and n large enough every *projective Steiner triple system* $PG(n, 2)$ (cf. [3]) satisfies the conditions of Theorem 1.2. In other words (for $k = 3$ and $l = 2$), projective Steiner triple systems $PG(n, 2)$ have the property of $(3, 2)$ -system \mathcal{T} with n sufficiently large.

Theorem 1.2, though quite powerful, gives no explicit estimate of the size of the graph \mathcal{T} . A quantitative extension of Theorem 1.2 is the main result of this note.

Let \mathcal{H} be an l -uniform hypergraph. Define the *clique number* of \mathcal{H} as

$$\omega(\mathcal{H}) = \max \{t \in \mathbb{N} \mid \mathcal{H} \supseteq K_t^{(l)}\}.$$

Let $\mathcal{G} = (V, \mathcal{E})$ be (k, l) -system. Define also the clique number for (k, l) -system as

$$\omega(\mathcal{G}, k, l) = \min \{\omega((V, S(\mathcal{E}))) \mid S \text{ is a selector on } \mathcal{G}\}.$$

Furthermore, let

$$\omega(n, k, l) = \max \{\omega(\mathcal{G}) \mid \mathcal{G} \text{ is } (k, l)\text{-system of order } n\}.$$

Theorem 1.2 states that for any fixed k and l the function $\omega(n, k, l) \rightarrow \infty$ as $n \rightarrow \infty$. For *partial Steiner triple systems* (PSTS), *i.e.*, where $k = 3$ and $l = 2$, we show the following explicit bounds.

Theorem 1.3.

$$(1 - o(1)) \log_2 \log_2 n \leq \omega(n, 3, 2) \leq 2 \log_3 n + 1.$$

2 Proof of Theorem 1.2

Let \leq be a linear ordering of vertices V . For a given hypergraph $\mathcal{G} = (V, \mathcal{E})$ denote by (\mathcal{G}, \leq) the hypergraph with linear ordering \leq on its vertices. Let (\mathcal{G}, \leq) and (\mathcal{H}, \leq) be two ordered hypergraphs with $\mathcal{G} = (V, \mathcal{E})$ and $\mathcal{H} =$

(W, \mathcal{F}) . Say the mapping $\phi : V \rightarrow W$ is an *ordered embedding* if for all $v < v'$, $v, v' \in V$, $\phi(v) < \phi(v')$, and $\{\phi(v_1), \phi(v_2), \dots, \phi(v_k)\} \in \mathcal{F}$ if and only if $\{v_1, v_2, \dots, v_k\} \in \mathcal{E}$.

We use the Ramsey theorem for Steiner systems established by J. Nešetřil and the third author.

Theorem 2.1 ([13]). *Let (\mathcal{G}, \leq) be an ordered k -uniform hypergraph such that $\mathcal{G} \in S(k, l)$. Let $r \geq 2$ be an integer. Then, there exists an ordered k -uniform hypergraph (\mathcal{H}, \leq) with $\mathcal{H} \in S(k, l)$ and such that for every partition of the edges $\mathcal{E}(\mathcal{H}) = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots \cup \mathcal{E}_r$ there exists i , $1 \leq i \leq r$, and an ordered embedding $\phi : V(\mathcal{G}) \rightarrow V((V, \mathcal{E}_i))$.*

Proof of Theorem 1.2. First we are going to define an ordered Steiner system (\mathcal{G}, \leq) to which we will apply Theorem 2.1. Let $[k] = \{1, \dots, k\}$. For each $L \in \binom{[k]}{l}$ consider an ordered set T_L , $|T_L| = t$ so that for $L \neq L'$, $T_L \cap T_{L'} = \emptyset$. Now for each T_L , where say $L = \{m_1 < \dots < m_l\}$, and every l -element subset $U = \{u_1 < \dots < u_l\} \subset T_L$ consider a k -element set $V(L, U) = \{v_1 < \dots < v_k\}$ such that:

- (i) $v_{m_i} = u_i$ for each $i = 1, \dots, l$, and
- (ii) $V(L, U) \cap V(L, U') \subset T_L$ for each L , and
- (iii) $V(L, U) \cap V(L', U') = \emptyset$, whenever $U \neq U'$.

Observe that (ii) and (iii) is equivalent to saying that the sets $V(L, U) \setminus U$ are pairwise disjoint for distinct U and L . For each $L \in \binom{[k]}{l}$ set

$$V_L = \bigcup \{V(L, U) \mid U \in \binom{T_L}{l}\}$$

and

$$V = \bigcup \{V_L \mid L \in \binom{[k]}{l}\}.$$

Let

$$\mathcal{E}_L = \{V(L, U) \mid U \in \binom{T_L}{l}\}, \tag{1}$$

and

$$\mathcal{E} = \bigcup \{\mathcal{E}_L \mid L \in \binom{[k]}{l}\}. \tag{2}$$

Clearly $\mathcal{G} = (V, \mathcal{E})$ is (k, l) -system. Let \leq be an arbitrary linear extension of the order we considered on elements of V . Let $r = \binom{k}{l}$ and let (\mathcal{H}, \leq) be a graph guaranteed by Theorem 2.1. We claim that \mathcal{T} is the desired graph \mathcal{H} .

Consider an arbitrary selector $\mathbb{S} : \mathcal{H} \rightarrow \binom{V(\mathcal{H})}{l}$ (for convenience we identify \mathcal{H} with its edge set). Consider the following partition of the edges of \mathcal{H} as

$$\mathcal{H} = \bigcup \{ \mathcal{H}_L \mid L \in \binom{[k]}{l} \},$$

where $\mathcal{H}_L = \{ E \in \mathcal{H} \mid E = \{x_1 < \dots < x_k\} \text{ and } \mathbb{S}(E) = \{x_i \mid i \in L\} \}$. By Theorem 2.1 there exists $L_0 \in \binom{[k]}{l}$ and a copy of (\mathcal{G}, \leq) in (\mathcal{H}, \leq) , say (\mathcal{G}_0, \leq) , such that $E(\mathcal{G}_0) \subset \mathcal{H}_{L_0}$. In particular, all edges $E = \{x_1 < \dots < x_k\}$ of $\mathcal{E}_{L_0} \subset E(\mathcal{G}_0)$ (cf. (1) and (2)) have the property that $\mathbb{S}(E) = \{x_i \mid i \in L_0\}$. Consequently, the set $T_{L_0} = \bigcup \{ \mathbb{S}(E) \mid E \in \mathcal{E}_{L_0} \}$ induces a clique $K_t^{(l)}$. \square

3 Proof of Theorem 1.3

First, we find an upper bound on $\omega(n, 3, 2)$ by using a simple probabilistic argument.

Proof of Theorem 1.3 (upper bound). We show that for sufficiently large n the following inequality holds:

$$\omega(n, 3, 2) \leq 2 \log_3 n + 1. \quad (3)$$

In order to prove (3), it suffices to show that $\omega(\mathcal{G}, 3, 2) \leq 2 \log_3 n + 1$ for any PSTS \mathcal{G} of order n . For a given PSTS $\mathcal{G} = (V, \mathcal{E})$ with $|V| = n$ we show that there exists a selector $\mathbb{S} : \mathcal{E} \rightarrow [V]^2$ for which $(M, \mathbb{S}(\mathcal{E}) \cap [M]^2)$ is not a complete graph, *i.e.*, $\mathbb{S}(\mathcal{E}) \cap [M]^2 \neq [M]^2$, whenever $M \subseteq V$ and $|M| > 2 \log_3 n + 1$. Let $\mathbb{S} : \mathcal{E} \rightarrow [V]^2$ be a *random selector* defined by $\Pr(\mathbb{S} = \mathbb{S}) = \frac{1}{\binom{3}{2}^{|\mathcal{E}|}}$ for every possible selector \mathbb{S} on \mathcal{G} . For a fixed set M with $|M| = m$ we have

$$\Pr(\mathbb{S}(\mathcal{E}) \cap [M]^2 = [M]^2) \leq 3^{-\binom{m}{2}},$$

since from $3^{|\mathcal{E}|}$ selectors at most $3^{|\mathcal{E}| - \binom{m}{2}}$ of them keep $[M]^2$ complete. Thus,

$$\Pr(\exists M \in [V]^m \text{ such that } \mathbb{S}(\mathcal{E}) \cap [M]^2 = [M]^2) \leq \binom{n}{m} 3^{-\binom{m}{2}},$$

and equivalently

$$\Pr(\forall M \in [V]^m \text{ we have } \mathbb{S}(\mathcal{E}) \cap [M]^2 \neq [M]^2) \geq 1 - \binom{n}{m} 3^{-\binom{m}{2}}. \quad (4)$$

Note that for $m > 2 \log_3 n + 1$ we get $\binom{n}{m} \leq n^m < \left(3^{\frac{m-1}{2}}\right)^m = 3^{\binom{m}{2}}$. Consequently, the right side of (4) is positive, *i.e.*, there exists a selector with the required property. \square

In order to prove the lower bound on $\omega(n, 3, 2)$ we need to show the existence of the PSTS with the property that any selector chooses a clique of size $\Omega(\ln \ln n)$. To this end, we construct a PSTS with the property that any sufficiently large subset of its vertices induces many triples. We need one auxiliary result, *i.e.*, Proposition 3.3, which follows from a special version of Lovász Local Lemma, *i.e.*, Corollary 3.2. Let A_1, \dots, A_n be events in a probability space. A graph $\Gamma = (V, E)$ on the set vertices $\{1, 2, \dots, n\}$ is called a *dependency graph* for the events A_1, \dots, A_n if for each i , $1 \leq i \leq n$, the event A_i is mutually independent of all the events $\{A_j \mid \{i, j\} \notin E\}$.

Lemma 3.1 (Lovász Local Lemma (see, *e.g.*, [2])). *Suppose that $\Gamma = (V, E)$ is a dependency graph for the events A_1, \dots, A_n and suppose there are real numbers x_1, \dots, x_n such that $0 < x_i < 1$ and $\Pr(A_i) \leq x_i \prod_{\{i, j\} \in E} (1 - x_j)$ for all $1 \leq i \leq n$. Then, $\Pr\left(\bigcap_{i=1}^n \bar{A}_i\right) > 0$, *i.e.*, with positive probability no event A_i holds.*

In the proof of Proposition 3.3 we will use the following consequence of Lemma 3.1.

Corollary 3.2. *(For a similar result see [19].) Let A_1, \dots, A_n be events with a dependency graph $\Gamma = (V, E)$. Suppose, there exist real numbers y_1, \dots, y_n, δ such that $0 < \delta < 1$, $0 < y_i \Pr(A_i) \leq \delta$ and $\sum_{\{i, j\} \in E} y_j \Pr(A_j) \leq (1 - \delta) \ln(y_i)$ for all $1 \leq i \leq n$. Then, $\Pr\left(\bigcap_{i=1}^n \bar{A}_i\right) > 0$.*

Proof. For each i , $1 \leq i \leq n$, set $x_i = y_i \Pr(A_i)$. Note that $0 < x_i < 1$ and

$$\begin{aligned} \prod_{\{i,j\} \in E} (1 - x_j) &= \prod_{\{i,j\} \in E} \left(1 - y_j \Pr(A_j)\right) \geq \prod_{\{i,j\} \in E} \exp\left(\frac{-y_j \Pr(A_j)}{1 - y_j \Pr(A_j)}\right) \\ &= \exp\left(-\sum_{\{i,j\} \in E} \frac{y_j \Pr(A_j)}{1 - y_j \Pr(A_j)}\right) \geq \exp\left(-\sum_{\{i,j\} \in E} \frac{y_j \Pr(A_j)}{1 - \delta}\right) \\ &\geq \exp(-\ln(y_i)) = \frac{1}{y_i} = \frac{\Pr(A_i)}{x_i}, \end{aligned}$$

and hence, the assumptions of Lemma 3.1 are satisfied. \square

Proposition 3.3. *There exists a positive constant c such that for any $\varepsilon > 0$ and any sufficiently large $n \geq n_0(\varepsilon)$ there exists a PSTS $\mathcal{G} = (V, \mathcal{E})$ with $|V| = n$ and with the property that whenever $M \subseteq V$ with $|M| = m \geq n^{\frac{1}{2} + \varepsilon}$, then $|\mathcal{E} \cap [M]^3| > \frac{c}{2n} \binom{m}{3}$.*

Proof. By a standard averaging argument it is enough to show that the statement holds for $m = \lceil n^{\frac{1}{2} + \varepsilon} \rceil$. Set $c = \frac{1}{102}$ and $\varepsilon > 0$ be given. Let $\mathcal{G} = (V, \mathbb{E})$ be a random 3-uniform hypergraph with vertex set V , $|V| = n$, and with hyperedges chosen independently with probability $p = \frac{c}{n}$. For $L \in [V]^4$ let A_L be the event that $|\mathbb{E} \cap [L]^3| \geq 2$, *i.e.*, there is a pair, which is contained in at least two triples. For $M \in [V]^m$ let B_M be the event that $|\mathbb{E} \cap [M]^3| \leq \frac{p}{2} \binom{m}{3}$. Note that for $L, \hat{L} \in [V]^4$, A_L is independent of all $A_{\hat{L}}$ with $|L \cap \hat{L}| < 3$. Similarly, for $M \in [V]^m$, B_M is independent of all A_L and $B_{\hat{M}}$ with $|M \cap L| < 3$ and $|M \cap \hat{M}| < 3$, respectively. Let

$$A = \bigcap_{L \in [V]^4} \bar{A}_L \cap \bigcap_{M \in [V]^m} \bar{B}_M.$$

If $\Pr(A) > 0$, then there is a 3-uniform hypergraph $\mathcal{G} = (V, \mathcal{E})$ which is a PSTS (no pair is covered more than once) such that for any $M \subseteq V$ we have $|\mathcal{E} \cap [M]^3| > \frac{p}{2} \binom{m}{3}$.

In order to complete the proof, we show that $\Pr(A) > 0$. According to Corollary 3.2 (applied with $\delta = \frac{1}{100}$), it suffices to find positive real numbers y_L and z_M , for all $L \in [V]^4$ and $M \in [V]^m$, so that

$$y_L \Pr(A_L) \leq \frac{1}{100}, \tag{5}$$

$$z_M \Pr(B_M) \leq \frac{1}{100}, \quad (6)$$

$$\sum_{|\hat{L} \cap L| \geq 3} y_{\hat{L}} \Pr(A_{\hat{L}}) + \sum_{|M \cap L| \geq 3} z_M \Pr(B_M) \leq \frac{99}{100} \ln(y_L), \quad (7)$$

and

$$\sum_{|L \cap M| \geq 3} y_L \Pr(A_L) + \sum_{|\hat{M} \cap M| \geq 3} z_{\hat{M}} \Pr(B_{\hat{M}}) \leq \frac{99}{100} \ln(z_M). \quad (8)$$

First, let us estimate $\Pr(A_L)$ and $\Pr(B_M)$. For each $L \in [L]^4$

$$\Pr(A_L) = \binom{4}{2} p^2 (1-p)^2 + \binom{4}{3} p^3 (1-p) + p^4 < 6p^2,$$

since $p \leq 1$. Hence, for every $L \in [V]^4$ we have

$$\Pr(A_L) \leq \frac{6c^2}{n^2}. \quad (9)$$

To estimate $\Pr(B_M)$ we will use Chernoff's inequality (see, *e.g.*, Theorem 2.1 in [11]). Let $X \sim B\left(\binom{m}{3}, p\right)$ be a random variable with binomial distribution. Then, $E[X] = \binom{m}{3} p$ and Chernoff's inequality yields

$$\Pr(B_M) = \Pr\left(X \leq \frac{1}{2} E[X]\right) \leq \exp\left(-\frac{1}{8} E[X]\right) = \exp\left(-\frac{1}{8} \frac{c}{n} \binom{m}{3}\right).$$

Hence, for sufficiently large n ,

$$\Pr(B_M) < \exp\left(-\frac{c}{50} \frac{m^3}{n}\right). \quad (10)$$

Now for every $L \in [V]^4$ define

$$y_L = 1 + \frac{1}{n},$$

and for every $M \in [V]^m$ define

$$z_M = \exp\left(\frac{c}{100} \frac{m^3}{n}\right).$$

For a given $L \in [V]^4$ and n large enough, we obtain by (9)

$$y_L \Pr(A_L) \leq \left(1 + \frac{1}{n}\right) \frac{6c^2}{n^2} \leq \frac{1}{100}.$$

Similarly, since $m = \lceil n^{\frac{1}{2} + \varepsilon} \rceil$, (10) yields for n large enough

$$z_M \Pr(B_M) \leq \exp\left(\frac{c}{100} \frac{m^3}{n}\right) \exp\left(-\frac{c}{50} \frac{m^3}{n}\right) = \exp\left(-\frac{c}{100} \frac{m^3}{n}\right) \leq \frac{1}{100}.$$

Thus, conditions (5) and (6) are satisfied. To complete the proof of Proposition 3.3 we need to show that conditions (7) and (8) are satisfied as well.

For a given $L \in [V]^4$ the number of \hat{L} 's such that $\hat{L} \in [V]^4$ and $|\hat{L} \cap L| \geq 3$ is $\binom{4}{3}(n-4) < 4n$, and the number of M 's such that $M \in [V]^m$ and $|L \cap M| \geq 3$ is trivially less than $\binom{n}{m} \leq \left(\frac{ne}{m}\right)^m$, where e denotes the base of the natural logarithmic function. Thus,

$$\begin{aligned} \sum_{|\hat{L} \cap L| \geq 3} y_{\hat{L}} \Pr(A_{\hat{L}}) &+ \sum_{|M \cap L| \geq 3} z_M \Pr(B_M) \\ &\leq 4n \left(1 + \frac{1}{n}\right) \frac{6c^2}{n^2} + \left(\frac{ne}{m}\right)^m \exp\left(\frac{c}{100} \frac{m^3}{n}\right) \exp\left(-\frac{c}{50} \frac{m^3}{n}\right) \\ &= \left(1 + \frac{1}{n}\right) \frac{24c^2}{n} + \exp\left(m \left(-\frac{c}{100} \frac{m^2}{n} + \ln\left(\frac{ne}{m}\right)\right)\right). \end{aligned} \quad (11)$$

Since $m = \lceil n^{\frac{1}{2} + \varepsilon} \rceil$, then for sufficiently large n we have $-\frac{c}{100} \frac{m^2}{n} + \ln\left(\frac{ne}{m}\right) \leq -1$. Hence, the second term of (11) can be estimated by

$$\exp\left(m \left(-\frac{c}{100} \frac{m^2}{n} + \ln\left(\frac{ne}{m}\right)\right)\right) \leq \exp(-m) \leq \exp(-\sqrt{n}),$$

which yields in (11) for sufficiently large n

$$\begin{aligned} &\left(1 + \frac{1}{n}\right) \frac{24c^2}{n} + \exp\left(m \left(-\frac{c}{100} \frac{m^2}{n} + \ln\left(\frac{ne}{m}\right)\right)\right) \\ &\leq \left(1 + \frac{1}{n}\right) \frac{24c^2}{n} + \exp(-\sqrt{n}) = \frac{99}{100} \frac{2400c^2}{99} \left(1 + \frac{1}{n}\right) \frac{1}{n} + \exp(-\sqrt{n}). \end{aligned} \quad (12)$$

One can check that for $a < 1$ and x positive and sufficiently small number $a(1+x)x < \ln(1+x)$. Applying this inequality with $a = \frac{2400c^2}{99} < 1$ (recall

$c = \frac{1}{102}$) yields that the right side of (12) can be bounded from above (for n large enough) by $\frac{99}{100} \ln\left(1 + \frac{1}{n}\right)$. Thus,

$$\sum_{|\hat{L} \cap L| \geq 3} y_{\hat{L}} \Pr(A_{\hat{L}}) + \sum_{|M \cap L| \geq 3} z_M \Pr(B_M) \leq (11) \leq (12) \leq \frac{99}{100} \ln(y_L),$$

which proves (7).

Similarly, we show that (8) also holds. For a given $M \in [V]^m$, the number of L 's such that $L \in [V]^4$ and $|L \cap M| \geq 3$ is at most $\binom{m}{3}(n-3) \leq \frac{m^3 n}{6}$. Again the number of \hat{M} 's such that $\hat{M} \in [V]^m$ and $|M \cap \hat{M}| \geq 3$ is trivially less than $\binom{n}{m} \leq \left(\frac{ne}{m}\right)^m$. Thus,

$$\begin{aligned} \sum_{|L \cap M| \geq 3} y_L \Pr(A_L) + \sum_{|\hat{M} \cap M| \geq 3} z_{\hat{M}} \Pr(B_{\hat{M}}) & \\ & \leq \frac{m^3 n}{6} \left(1 + \frac{1}{n}\right) \frac{6c^2}{n^2} + \left(\frac{ne}{m}\right)^m \exp\left(\frac{c}{100} \frac{m^3}{n}\right) \exp\left(-\frac{c}{50} \frac{m^3}{n}\right) \\ & = \left(1 + \frac{1}{n}\right) \frac{c^2 m^3}{n} + \exp\left(m\left(-\frac{c}{100} \frac{m^2}{n} + \ln\left(\frac{ne}{m}\right)\right)\right) \\ & \leq \left(1 + \frac{1}{n}\right) \frac{c^2 m^3}{n} + \exp(-\sqrt{n}). \end{aligned} \quad (13)$$

Since $c < \frac{99}{10000}$ (recall $c = \frac{1}{102}$), then for n large enough $(1 + \frac{1}{n})c < \frac{99}{10000}$ as well. Consequently,

$$\sum_{|L \cap M| \geq 3} y_L \Pr(A_L) + \sum_{|\hat{M} \cap M| \geq 3} z_{\hat{M}} \Pr(B_{\hat{M}}) \leq (13) \leq \frac{99}{100} \frac{c}{100} \frac{m^3}{n} = \frac{99}{100} \ln(z_M).$$

This completes the proof of Proposition 3.3. \square

Proof of Theorem 1.3 (lower bound). We show that for sufficiently large n the following inequality holds:

$$(1 - o(1)) \log_2 \log_2 n \leq \omega(n, 3, 2). \quad (14)$$

Let c, ε (with $\frac{1}{2} > \varepsilon > 0$), and n_0 be from Proposition 3.3. Let c_1 be a positive constant such that $\frac{c}{2n} \binom{m}{3} \geq c_1 \frac{m^3}{n}$, for $n \geq n_0$ and $n \geq m \geq n^{\frac{1}{2} + \varepsilon}$.

Proposition 3.3 guarantees the existence of a PSTS $\mathcal{G} = (V, \mathcal{E})$ with $|V| = n$, which satisfies

$$|\mathcal{E} \cap [M]^3| > \frac{c}{2n} \binom{m}{3} \geq c_1 \frac{m^3}{n}, \quad (15)$$

whenever $M \subseteq V$ and $|M| = m \geq n^{\frac{1}{2}+\varepsilon}$ (since $c = \frac{1}{102}$ works in Proposition 3.3, $c_1 = \frac{1}{1250}$ satisfies (15)). Let $\mathbf{S} : \mathcal{E} \rightarrow [V]^2$ be a selector on \mathcal{G} . Then, for any $M \subseteq V$ with $m \geq n^{\frac{1}{2}+\varepsilon}$ the number of edges $\mathbf{S}(\mathcal{E})$ induced on the set M is at least $|\mathcal{E} \cap [M]^3|$. Hence,

$$|\mathbf{S}(\mathcal{E}) \cap [M]^2| \geq |\mathcal{E} \cap [M]^3| \geq c_1 \frac{m^3}{n}. \quad (16)$$

We construct a clique of size $\log_2 \log_2 n - O(1)$. Set $M_1 = V$. Since $|M_1| = n \geq n^{\frac{1}{2}+\varepsilon}$, (16) yields that

$$|\mathbf{S}(\mathcal{E}) \cap [M_1]^2| \geq c_1 \frac{n^3}{n} = c_1 n^2.$$

Consequently, there must be an element $a_1 \in M_1$ and a set $M_2 \subseteq M_1$ with $|M_2| \geq \frac{2c_1 n^2}{n} = 2c_1 n$ such that $\{a_1, x\} \in \mathbf{S}(\mathcal{E})$ for any $x \in M_2$.

Set $c_2 = 2c_1$. Then, $|M_2| \geq c_2 n$. If $c_2 n \geq n^{\frac{1}{2}+\varepsilon}$, then (16) infers that

$$|\mathbf{S}(\mathcal{E}) \cap [M_2]^2| \geq c_1 \frac{(c_2 n)^3}{n} = c_1 c_2^3 n^2.$$

Thus, there must be an element $a_2 \in M_2$ and a set $M_3 \subseteq M_2$ with $|M_3| \geq \frac{2c_1 c_2^3 n^2}{c_2 n} = 2c_1 c_2^2 n$ such that $\{a_2, x\} \in \mathbf{S}(\mathcal{E})$ for any $x \in M_3$.

In general, set $c_{i+1} = 2c_1 c_i^2$, which leads to $c_{i+1} = (2c_1)^{2^i-1} = \left(\frac{1}{625}\right)^{2^i-1}$. We can carry on with this construction as long as $c_i n > n^{\frac{1}{2}+\varepsilon}$. If i_0 is the largest such i , then $c_{i_0} n = \Theta(n^{\frac{1}{2}+\varepsilon})$ or equivalently $625^{(2^{i_0}-1)} = \Theta(n^{\frac{1}{2}-\varepsilon})$, which yields $i_0 \geq \log_2 \log_2 n - O(1)$. □

4 Concluding remarks

Our main tool to find the lower bound on $\omega(n, 3, 2)$ was Proposition 3.3. In particular, for a given PSTS $\mathcal{G} = (V, \mathcal{E})$, a selector \mathbf{S} , and a set $M \subseteq V$,

$|M| > n^{\frac{1}{2}+\varepsilon}$, we concluded in (16) that the number of edges $\mathbf{S}(\mathcal{E})$ induced on the set M is at least $|\mathcal{E} \cap [M]^3|$. However, it looks very likely that in general this number, *i.e.*, $|\mathbf{S}(\mathcal{E}) \cap [M]^2|$, is much bigger. In fact, there are many edges in $\mathbf{S}(\mathcal{E}) \cap [M]^2$, which are contained in triples that do not lie entirely in M . We conjecture that the right magnitude of $\omega(n, 3, 2)$ is $\log_2(n)$.

Conjecture 4.1. *There exists a constant c such that*

$$c \log_2(n) \leq \omega(n, 3, 2).$$

We believe that our proof of Theorem 1.3 can be modified to give similar bounds on $\omega(n, k, 2)$. The problem of estimating $\omega(n, k, l)$, $l \geq 3$, seems to be however harder.

5 Acknowledgment

We would like to thank the referee for his or her very valuable comments.

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