On Minimum Saturated Matrices

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Abstract

Motivated by the work of Anstee, Griggs, and Sali on forbidden submatrices and the extremal sat-function for graphs, we introduce sat-type problems for matrices. Let $\mathcal{F}$ be a family of k-row matrices. A matrix M is called $\mathcal{F}$-*admissible* if M contains no submatrix $F \in \mathcal{F}$ (as a row and column permutation of F). Moreover, M is $\mathcal{F}$-*saturated* if M is $\mathcal{F}$-admissible but the addition of any column not present in M violates this property. In this paper we consider the function $\mathrm{sat}(n, \mathcal{F})$ which is the *minimum* number of columns of an $\mathcal{F}$-saturated matrix with n rows. We establish the estimate $\mathrm{sat}(n, \mathcal{F}) = O(n^{k-1})$ for any family $\mathcal{F}$ of k-row matrices and also compute the sat-function for a few small forbidden matrices.

1 Introduction

First, we introduce some notation. Let the shortcut 'an $m \times n$-matrix' M mean a matrix with m rows (which we view as horizontal arrays) and n 'vertical' columns. Usually, we restrict entries to only two values, 0 and 1; in all other cases, we indicate the range explicitly. For an $n \times m$-matrix M, its *order* $v(M) = n$ is the number of rows and its *size* $e(M) = m$ is the number of columns. We distinguish the expressions like 'an n-row matrix' and 'an n-row' standing respectively for a matrix with n rows and for a row containing n elements.

For an $n \times m$-matrix M and sets $A \subseteq [n]$ and $B \subseteq [m]$, $M(A, B)$ is the $|A| \times |B|$-submatrix of M formed by the rows indexed by A and the columns indexed by B. We use the following obvious shorthands: $M(A,) = M(A, [m])$, $M(A, i) = M(A, \{i\})$, etc. For example, the rows and the columns of M are denoted by $M(1,), \ldots, M(n,)$ and $M(, 1), \ldots, M(, m)$ respectively while individual entries – by $M(i, j)$, $i \in [n]$, $j \in [m]$.

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A matrix F is a *submatrix* of a matrix A (denoted as $F \subseteq A$) if deleting some set of rows and columns of A we can obtain a submatrix which is a row/column permutation of F. The *transpose* of M is denoted by M^T (which we use mostly to denote vertical columns for typographical reasons); $(a)^i$ is the (horizontal) sequence containing element a i times. The $n \times (m_1 + m_2)$-matrix $[M_1, M_2]$ is obtained by concatenating an $n \times m_1$-matrix M_1 and an $n \times m_2$-matrix M_2. We write $N \cong M$ to say that N is a column/row permutation of M. Thus, $N \subseteq M$ if $N \cong M(A, B)$ for some index sets A and B. The *complement* $1 - M$ of a matrix M is obtained by interchanging ones and zeros in M. The *characteristic function* χ_Y of $Y \subseteq [n]$ is the n-column with ith entry being 1 if $i \in Y$ and 0 otherwise.

Many interesting and important properties of matrices (that is, classes of matrices) can be defined by listing forbidden submatrices. (Some authors use the term 'forbidden configurations' then.) More precisely, given a family $\mathcal{F}$ of matrices (referred to as *forbidden*), we say that a matrix M is *$\mathcal{F}$-admissible* (or *$\mathcal{F}$-free*) if M contains no $F \in \mathcal{F}$ as a submatrix. A *simple matrix* M (that is, a matrix without repeated columns) is called *$\mathcal{F}$-saturated* or (*$\mathcal{F}$-critical*) if M is $\mathcal{F}$-free but the addition of any column not present in M violates this property; this is denoted by $M \in \mathrm{SAT}(n, \mathcal{F})$, $n = v(M)$. Note that, although the definition requires that M is simple, we allow multiple columns in matrices belonging to $\mathcal{F}$.

One well-known extremal problem is to consider $\mathrm{forb}(n, \mathcal{F})$, the maximum size of a simple $\mathcal{F}$-free matrix with n rows or, equivalently, the maximal size of $M \in \mathrm{SAT}(n, \mathcal{F})$. Many different results on the topic have been obtained, some of them will be mentioned in the due course, but we do not even try to give a comprehensive survey here. We would only mention a remarkable fact that one of the first forb-type results, namely formula (1) here, proved independently by Vapnik and Chervonenkis [21], Perles and Shelah [19], and Sauer [18], was motivated by such different topics as probability, logic and a problem of Erdős on infinite set systems.

The forb-problem is reminiscent of the Turán function $\mathrm{ex}(n, \mathcal{F})$: given a family $\mathcal{F}$ of forbidden graphs, find the maximum size of an $\mathcal{F}$-free graph on n vertices not containing any member of $\mathcal{F}$ as a subgraph (see e.g. surveys [15, 20]). Erdős, Hajnal, and Moon [11] considered the 'dual' function $\mathrm{sat}(n, \mathcal{F})$, the *minimum* size of a maximal $\mathcal{F}$-free graph on n vertices. This is an active area of extremal graph theory, see [6, Section 3] for a short overview.

Here we consider the 'dual' of the forb-problem for matrices. Namely, we are interested in the value of $\mathrm{sat}(n, \mathcal{F})$, the *minimum* size of an $\mathcal{F}$-saturated matrix with n rows:

$$\mathrm{sat}(n, \mathcal{F}) = \min\{e(M) : M \in \mathrm{SAT}(n, \mathcal{F})\}.$$

We decided to use the same notation as for its graph counterpart. This should not cause any confusion as this paper will deal with matrices alone.

Obviously, $\mathrm{sat}(n, \mathcal{F}) \leq \mathrm{forb}(n, \mathcal{F})$. If $\mathcal{F} = \{F\}$ consists of a single forbidden matrix F then we write $\mathrm{SAT}(n, F) = \mathrm{SAT}(n, \{F\})$, etc.

A simple n-row matrix M is *monotonically $\mathcal{F}$-saturated*, denoted $M \in \text{m-SAT}(n, \mathcal{F})$, if the addition of any new n-column C to M *creates* a new forbidden submatrix, that is, there is $A \subseteq [n]$ such that $M(A,)$ is $\mathcal{F}$-free while $[M, C](A,)$ is not. Clearly, $\text{m-SAT}(n, \mathcal{F}) \supseteq \text{SAT}(n, \mathcal{F})$, so $\text{m-sat}(n, \mathcal{F}) \leq \text{sat}(n, \mathcal{F})$ where as always $\text{m-sat}(n, \mathcal{F}) = \min\{e(M) : M \in \text{m-SAT}(n, \mathcal{F})\}$. This is the natural analog of the m-sat-function for graphs, which was studied already in [11]. It is useful in proving lower bounds on $\text{sat}(n, \mathcal{F})$ via induction on n.

There is an obvious generalization of these problems when we consider the class of $[0, l]$-*matrices* (matrices whose entries can assume $(l+1)$-values from $\{0, \ldots, l\}$) with the above definitions going practically unchanged. In this case we will use symbols like $\text{SAT}(n, \mathcal{F}; l)$, etc., whereas the default $\text{SAT}(n, \mathcal{F}) = \text{SAT}(n, \mathcal{F}; 1)$ is the usual notion.

By T_k^l we denote the simple $k \times \binom{k}{l}$-matrix consisting of all k-columns with exactly l ones and by K_k–the $k \times 2^k$ matrix of all possible columns of size k. Naturally, $T_k^{\leq l}$ denotes the $k \times f(k, l)$-matrix consisting of all distinct columns with at most l ones, etc, where we use the shortcut

$$f(n, k) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}.$$

Vapnik and Chervonenkis [21], Perles and Shelah [19], and Sauer [18] showed independently that

$$\text{forb}(n, K_k) = f(n, k-1). \tag{1}$$

As formula (1) plays an important role in our consideration of the sat-function, we give a proof, for the sake of completeness, of a more general result which is of independent interest. Namely, in the class of $[0, l]$-matrices, we compute the maximum size of a K_k^l-free n-row matrix, where K_k^l is the k-row matrix of size $(l+1)^k$ consisting of all distinct k-columns made of symbols from $[0, l] = \{0, \ldots, l\}$. Our proof is based on the ideas of Frankl introduced in [13].

Theorem 1.1 *For any $l \geq 1$, $k \geq 1$ and $n \geq k-1$,*

$$\text{forb}(n, K_k^l; l) = \sum_{i=0}^{k-1} l^{n-i} \binom{n}{i}. \tag{2}$$

Proof. To prove the lower bound, consider the matrix made of all columns containing l at most $k-1$ times which is obviously K_k^l-admissible.

It remains to prove the upper bound. Consider any K_k^l-admissible matrix M of order n and size m. Fix an integer i, $1 \leq i \leq n$. Divide the set of columns of M into disjoint sets $\mathcal{C}_1, \ldots, \mathcal{C}_p$, $1 \leq p \leq m$, such that all columns in $\mathcal{C}_j$, $1 \leq j \leq p$, agree except in the ith row. Clearly, $|\mathcal{C}_j| \leq l+1$. Let $\mathcal{C}'_j$ be obtained from $\mathcal{C}_j$ by replacing the set of entries corresponding to the ith row in columns of $\mathcal{C}_j$ by $\{0, \ldots, |\mathcal{C}_j| - 1\}$. Let M'_i be a matrix consisting of all columns from $\mathcal{C}'_j$ over all $1 \leq j \leq p$.

Claim. If $M'_i \supseteq K_k^l$ then $M \supseteq K_k^l$.

Suppose $M'_i \supseteq K^k_l$. Let R be the subset of k rows of M'_i such that $M'_i(R,) \supseteq K^l_k$. If i is not in R then M also contains K^l_k (since M and M'_i are identical apart from row i). Thus, we may assume that i is in R. Let C be any column of K^k_l. Let C^l be the same column with l in the ith entry. Then some column D^l of M'_i gives C^l when restricted to R. And D^l has an l in the ith entry, so if D^l is in M'_i, D^l is also in M, and every column obtained from D^l by putting anything in the ith entry is also in M. But then C is a column of M when restricted to R. Thus $M \supseteq K^l_k$, as required.

In order to prove the theorem we recursively repeat the claim for every row i, $1 \le i \le n$, obtaining a matrix M' of size m. Observe that if any column of M' has k entries equal to l then $M' \supseteq K^l_k$ and so by the claim $M \supseteq K^l_k$, a contradiction. Hence, every column of M' has at most $k-1$ entries equal to l which gives the theorem. $\square$

2 General Results

Here we present some results dealing with $\mathrm{sat}(n, \mathcal{F})$ for a general family $\mathcal{F}$.

The following simple observation may be useful in tackling these problems. Let M' be obtained from $M \in \mathrm{SAT}(n, \mathcal{F})$ by *duplicating* the nth row of M, that is, we let $M'([n],) = M$ and $M'(n+1,) = M(n,)$. Suppose that M' is $\mathcal{F}$-admissible. Complete M', in an arbitrary way, to an $\mathcal{F}$-saturated matrix. Let C be any added $(n+1)$-column. As both $M'([n],)$ and $M'([n-1] \cup \{n+1\},)$ are equal to $M \in \mathrm{SAT}(n, \mathcal{F})$, we conclude that both $C([n])$ and $C([n-1] \cup \{n+1\})$ must be columns of M. As C is not an M'-column, $C = (C', b, 1-b)$ for some $(n-1)$-column C' and $b \in \{0, 1\}$ such that both $(C', 0)$ and $(C', 1)$ are columns of M. This implies that $\mathrm{sat}(n+1, \mathcal{F}) \le e(M) + 2d$, where d is the number of pairs of equal columns in M after we delete the nth row.

The above argument works for the m-sat-function. Namely, duplicate a row of some $M \in \text{m-SAT}(n, \mathcal{F})$ and add one by one missing columns so that no new forbidden submatrix appears; clearly the resulting matrix is monotonically $\mathcal{F}$-saturated and, like above, we add at most $2d$ extra rows. In particular, the following theorem follows.

Theorem 2.1 *Either* $\text{m-sat}(n, \mathcal{F})$ *is constant for large* n *or* $\text{m-sat}(n, \mathcal{F}) \ge n + 1$ *for every* n*. The analogous statement is true for the* sat*-function if no matrix in* $\mathcal{F}$ *has two equal rows.*

Proof. If we have some $M \in \text{m-SAT}(n, \mathcal{F})$ with at most n columns then a well-known theorem of Bondy [7] (see, *e.g.*, Theorem 2.1 in [5]) implies that there is $i \in [n]$ such that the removal of the ith row does not create two equal columns. Hence, the duplication of the ith row gives a monotonically $\mathcal{F}$-saturated matrix, which implies $\text{m-sat}(n+1, \mathcal{F}) \le \text{m-sat}(n, \mathcal{F})$ and the first part follows. The same argument establish the second part as the duplication of a row cannot create a forbidden submatrix by the condition on $\mathcal{F}$. $\square$

Suppose that $\mathcal{F}$ consists of k-row matrices. Is there any good general upper bound on $\mathrm{forb}(n, \mathcal{F})$ or $\mathrm{sat}(n, \mathcal{F})$? There were different papers dealing with general upper bounds on $\mathrm{forb}(n, \mathcal{F})$, *e.g.*, by Anstee and Füredi [4], by Frankl, Füredi and Pach [14] and by Anstee [1], until the conjecture of Anstee and Füredi [4] that $\mathrm{forb}(m, \mathcal{F}) = O(n^k)$ for any fixed $\mathcal{F}$ was elegantly proved by Füredi (see [2] for a proof).

On the other hand, we can show that $\mathrm{sat}(n, \mathcal{F}) = O(n^{k-1})$ for any family $\mathcal{F}$ of k-row matrices (including infinite families). This is the matrix analog of the main result in [17]. Note we cannot decrease the exponent of $k-1$ with the estimate remaining true for any $\mathcal{F}$; for example, $\mathrm{sat}(n, T_k^k) = f(n, k-1)$.

Theorem 2.2 *For any family $\mathcal{F}$ of k-row matrices,* $\mathrm{sat}(n, \mathcal{F}) = O(n^{k-1})$.

Proof. We may assume that K_k is $\mathcal{F}$-admissible (*i.e.* every matrix of $\mathcal{F}$ contains a pair of equal columns) for otherwise we are home by (1) as then $\mathrm{sat}(n, \mathcal{F}) \le \mathrm{forb}(n, K_k) = O(n^{k-1})$.

Let $l \in [0, k]$ be the smallest number such that there exists m for which $[mT_k^{\le l}, T_k^{>l}]$ is not $\mathcal{F}$-admissible. (Clearly, such l exists as $T_k^{\le k} = K_k$ is the complete matrix.) Let d be the maximal integer such that $[mT_k^{<l}, dT_k^l, T_k^{>l}]$ is $\mathcal{F}$-admissible for any m. Note that $d \ge 1$ as $[mT_k^{<l}, T_k^l, T_k^{>l}] = [mT_k^{<l}, T_k^{\ge l}]$ cannot contain a forbidden submatrix by the choice of l. Choose minimal m such that $[mT_k^{<l}, (d{+}1)T_k^l, T_k^{>l}]$ is not $\mathcal{F}$-admissible.

Suppose first that $l < k$. Given n, let N be the n-row matrix corresponding to the following set system:

$$H = \bigcup_{j \in [d-1]} \{Y \in [n]^{(l+1)} : \textstyle\sum_{y \in Y} y \equiv j \pmod{n}\}.$$

Here $X^{(i)} = \{Y \subseteq X : |Y| = i\}$ denotes the set of all subsets of X of size i.

Note that any $A \in [n]^{(l)}$ is covered by at most $d-1$ edges of H as there are at most $d-1$ possibilities to choose $i \in [n] \setminus A$ so that $A \cup \{i\} \in H$: namely, $i \equiv j - \sum_{a \in A} a \pmod{n}$ for $j \in [d-1]$.

On the other hand, the set H' of all l-subsets of $[n]$ covered by fewer than $d-1$ edges of H has size at most $2(d-1)\binom{n}{l-1}$. Indeed, if $A \in H'$ then, for some $j \in [d-1]$ and $x \in A$, $2x = j - \sum_{a \in A \setminus \{x\}} a \pmod{n}$ so, once $A \setminus \{x\}$ and j have been chosen, there are at most 2 choices for x.

Call $X \in [n]^{(k)}$ *bad* if, for some $A \in X^{(l)}$,

$$|\{Y \in H : Y \supseteq A,\ Y \cap (X \setminus A) = \emptyset\}| \le d-2. \tag{3}$$

To obtain a bad k-set X, we either complete some $A \in H'$ to any k-set or take any l-set A and let $X \supseteq A$ intersect some H-edge covering A. Therefore, the number of bad sets is at most

$$2(d-1)\binom{n}{l-1}\binom{n}{k-l} + \binom{n}{l}(d-1)\binom{n}{k-l-1} = O(n^{k-1}).$$

Let $M' = [N, T_n^l]$. Clearly,

$$M'(X,) \subseteq [\tbinom{n}{l} T_k^{<l}, dT_k^l, T_k^{l+1}]. \quad \text{for any } X \in [n]^{(k)}.$$

Hence, M' cannot contain a forbidden submatrix. Now complete it to arbitrary $M = [M', M''] \in \mathrm{SAT}(n, \mathcal{F})$.

Suppose that $e(M'') \neq O(n^{k-1})$. Then, by (1), $K_k \cong M''(X, Y)$ for some X, Y. Now, remove the columns corresponding to Y from M'' and repeat the procedure as long as possible to obtain $> O(n^{k-1})$ column-disjoint copies of K_k in M''. If some $X \in [n]^{(k)}$ appears more than d times then (because $T_n^l(X,) \supseteq mT_k^{<l}$ for $n \geq m + k$) $M(X,) \supseteq [mT_k^{<l}, (d+1)K_k]$ is not $\mathcal{F}$-admissible. Otherwise, $K_k \subseteq M''(X,)$ for some *good* (*i.e.,* not bad) $X \in [n]^{(k)}$; but then $N(X,) \supseteq (d-1)T_k^l$ and $M(X,)$ contains a forbidden matrix. This contradiction proves the required bound for $l < k$.

Suppose that $l = l(\mathcal{F})$ equals k; the above argument does not work in this case because $M' \supseteq T_n^l$ is too large. Let N_s denote the $(d+s) \times d$-matrix made of $\chi_{[s+d]\setminus\{i\}}$, $i \in [s+1, s+d]$, where $d = d(\mathcal{F})$ is as above; clearly, each N_s is $\mathcal{F}$-free. Also let C^* be obtained from a column C by adding d zeros to the end; to obtain M^*, we apply this operation to every column of a matrix M; this increases the number of rows by d.

Let $\mathcal{F}^*$ consist of all matrices M such that $[M^*, N_{v(M)}]$ is not $\mathcal{F}$-free; clearly, this property is not affected by a row/column permutation of M.

Note that $T = [mT_k^{<k}, T_k^k]$ is not $\mathcal{F}^*$-admissible as

$$[T^*, N_k]([k],) = [T, dT_k^k] \supseteq [mT_k^{<k}, (d+1)T_k^k]$$

so $l(\mathcal{F}^*) < k$ and, by the above argument, we can find $L \in \mathrm{SAT}(n-d, \mathcal{F}^*)$ with $O(n^{k-1})$ columns. Now, $M' = [L^*, N_{n-d}]$ is $\mathcal{F}$-free so complete it to an arbitrary $M \in \mathrm{SAT}(n, \mathcal{F})$. Let C be any added column; as $[M', C]([n-d],)$ is $\mathcal{F}^*$-free (otherwise M contains a forbidden submatrix), $C([n-d])$ either is a column of L or equals $((1)^{n-d})^T$. Hence, $e(M) \leq 2^d(e(M_1) + 1)$ and the theorem follows. $\square$

3 Forbidding Complete Matrices

Let us investigate the value of $\mathrm{sat}(n, K_k)$. (Recall that K_k is the $k \times 2^k$-matrix consisting of all distinct k-columns.) We are able to settle the cases $k = 2$ and $k = 3$.

We will use the following trivial lemma a couple of times.

Lemma 3.1 *Each row of any $M \in$ m-SAT(n, K_k), $n \geq k$, contains at least l ones and at least l zeros, $l = 2^{k-1} - 1$.*

Proof. Suppose that the first row $M(1,)$ has m_0 zeros followed by m_1 ones with $m_0 \geq m_1$ and $l > m_1$.

For $i \in [m_0]$, let C_i equal the ith column of M with the first entry 0 replaced by 1. Then $M' = [M, C_i]$ cannot contain K_k because the first row of M' contains too few 1's while $C_i([2, n])$ is a column of $M([2, n],)$. Therefore, C_i must be a column of M and $m_0 = m_1$.

But then M has at most $2^k - 2$ columns, which is a contradiction. $\square$

The following result sounds as a rather natural (and not difficult) question if reformulated in the terms of set systems but we have not been able to find it in the literature.

Theorem 3.2 *For* $n \geq 1$, $\text{m-sat}(n, K_2) = \text{sat}(n, K_2) = \text{forb}(n, K_2) = n + 1$.

Proof. Suppose that the statement is not true, that is, there exists a monotonically K_2-saturated matrix with its size not exceeding its order. By Theorem 2.1, $\text{m-sat}(n, K_2)$ is eventually constant so we can find an $n \times m$-matrix $M \in \text{m-SAT}(n, K_2)$ having two equal rows for some $n \in \mathbb{N}$.

As we are free to complement and permute rows, we may assume that, for some $i \geq 2$, $M(1,) = \cdots = M(i,)$ while $M(j,) \neq M(1,)$ and $M(j,) \neq 1 - M(1,)$ for any $j \in [i + 1, n]$. Note that $i < n$ as we do not allow multiple columns in M (and $m \geq v(K_2) - 1 = 3$). By permuting the rows in $M([i + 1, n],)$, we may assume that for some $l \in [i, n]$, we have $M(\{1, j\},) \supseteq K_2$ if and only if $j \in [i + 1, l]$.

Let $j \in [i + 1, n]$. By Lemma 3.1, the jth row $M(j,)$ has both entries 0 and 1. By the definition of i, $M(j,)$ is not equal to $M(1,)$ nor to $1 - M(1,)$. It easily follows that there are $f_j, g_j \in [m]$ with $M(1, f_j) = M(1, g_j)$, $M(j, f_j) \neq M(j, g_j)$; furthermore, we can find $h_j \in [m]$ with $M(1, h_j) = 1 - M(1, f_j)$. Let $b_j = M(j, h_j)$; we may assume that $M(j, g_j) = b_j$. Furthermore, for $j \in [i + 1, l]$, we choose f_j, g_j, h_j so that $b_j = M(j, 1)$, which is possible as $M(\{1, j\},) \supseteq K_2$.

Now, as $M \in \text{SAT}(n, K_2)$, the addition of the column

$$C = (1, (0)^{i-1}, b_{i+1}, \ldots, b_n)^T$$

(which is not in M because $C(1) \neq C(2)$) must create a new K_2-submatrix, say in the xth and yth rows, some $1 \leq x < y \leq m$. Clearly, $\{x, y\} \not\subseteq [i]$ because each column of $M([i], [m])$ is either $((0)^i)^T$ or $((1)^i)^T$. Also, it is impossible that $x \in [i]$ and $y \in [i + 1, n]$ because then, for some $a_1, a_2 \in [m]$, $M(y, a_1) = M(y, a_2) = 1 - C(y) = 1 - b_y$, $M(1, a_1) = 1 - M(1, a_2)$ and we can see that K_2 is isomorphic to $M(\{x, y\}, \{a_1, a_2, g_y, h_y\})$, which contradicts $K_2 \not\subseteq M(\{x, y\},)$. So we have to assume that $i < x < y \leq n$. As $M([i + 1, l], 1) = C([i + 1, l])$, $y > l$.

As $K_2 \not\subseteq M(\{x, y\},)$, no column of $M(\{x, y\}, [m])$ can equal $C(\{x, y\}) = (b_x, b_y)^T$; in particular, $M(y, g_x) = M(y, h_x) = 1 - b_y$ (as $M(x, g_x) = M(x, h_x) = b_x$). But then

$$K_2 \cong M(\{1, y\}, \{g_x, h_x, g_y, h_y\}), \tag{4}$$

which is a contradiction proving our theorem. □

Theorem 3.2 yields that $\text{sat}(n, K_2) = \text{forb}(n, K_2) = n + 1$ which, in our opinion, is rather surprising. A greater surprise is yet to come as we are going to show now that $\text{sat}(n, K_3)$ is constant for $n \geq 4$.

Theorem 3.3 *For* K_3 *the following holds:*

$$\text{sat}(n, K_3) = \begin{cases} 7, & \text{if } n = 3, \\ 10, & \text{if } n \geq 4. \end{cases}$$

Proof. The claim is trivial for $n = 3$, so assume $n \geq 4$. A computer search [10] revealed that

$$\mathrm{sat}(4, K_3) = \mathrm{sat}(5, K_3) = \mathrm{sat}(6, K_3) = \mathrm{sat}(7, K_3) = 10, \tag{5}$$

which suggested that $\mathrm{sat}(n, K_3)$ is constant. An example of a K_3-saturated 6×10-matrix is the following.

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

It is possible (but very boring) to check by hand that M is indeed K_3-saturated as is, in fact, any $n \times 10$-matrix M' obtained from M by duplicating any row, *cf.* Theorem 2.1. (The symmetries of M shorten the verification.) A K_3-saturated 5×10-matrix can be obtained from M by deleting one row (any). For $n = 4$, we have to provide a special example:

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

So $\mathrm{sat}(n, K_3) \leq 10$ for each $n \geq 4$ and, to prove the theorem, we have to show that no K_3-saturated matrix M with at most 9 columns and at least 4 rows can exist. Let us assume the contrary.

Claim 1. Any row of $M \in \mathrm{SAT}(n, K_3)$ necessarily contains at least four 0's and at least four 1's, for $n \geq 4$.

Suppose, on the contrary to the claim, that the first row $M(1,)$ contains only 3 zeros, say in the first three columns. (By Lemma 3.1 we must have at least 3 zeros.)

If we replace the ith of these zeros by 1, $i \in [3]$, then the obtained column C_i, if added to M, does not create any K_3-submatrix. Indeed, the first row of $[M, C_i]$ contains at most three 0's while $C_i([2, n])$ is a column of $M([2, n],)$. As M is K_3-saturated, C_1, C_2 and C_3 are columns of M. These columns differ only in the first entry from $M(, 1)$, $M(, 2)$ and $M(, 3)$ respectively. Therefore, for each $A \in [2, n]^{(3)}$, the matrix $M(A,)$ can contain at most $e(M) - 3 \leq 6$ distinct columns. But then any column C which is not the column of M and has the leading entry 1 (C exists as $n \geq 4$), contradicts the K_3-saturation of M as the first row of $[M, C]$ contains at most 3 zeros. This contradiction proves Claim 1.

Therefore, $e(M)$ is either 8 or 9. As we are free to complement the rows, we may assume that each row of M contains exactly four 1's. Call $A \in [n]^{(3)}$ (and also $M(A,)$) *nearly complete* if $M(A,)$ has 7 distinct columns.

Claim 2. Any nearly complete $M(A,)$ contains $(0,0,0)^T$ as a column.

Indeed, otherwise $M(A,) \supseteq T_3^{\geq 1}$ which already contains four 1's in each row; this implies that the (one or two) remaining columns must contain zeros only. This implies that $M(A,) \supseteq K_3$, which is a contradiction.

Claim 3. Every nearly complete $M(A,)$ contains T_3^1 as a submatrix.

Indeed, if $(0,0,1)^T$ is the missing column of $M(A,)$, then some 7 columns span a copy of $K_3 \setminus (0,0,1)^T$. By counting 1's in the rows we deduce that the remaining column(s) must have exactly one non-zero entry and one of them equals $(0,0,1)^T$, which is a contradiction.

Now fix any nearly complete A which exists by the K_3-saturation of M. Assume that $A = [3]$ and that the first 7 columns of $M([3],)$ are distinct. We know that the 3-column missing from $M([3],[7])$ has at least two 1's.

If $(1,1,1)^T$ is missing, then $M([3],[7])$ contains exactly three ones in each row, so the remaining column(s) of M must contain an extra 1 in each row. As $(1,1,1)^T$ is the missing column, we conclude that $e(M) = 9$ and the 8th and 9th columns of $M([3],)$ are, up to a row permutation, $(0,0,1)^T$ and $(1,1,0)^T$. This implies that $M([3],)$ contains the column $(0,0,0)^T$ only once. On the other hand, by Claims 2 and 3, M must contain the columns $((0)^n)^T$ and $((0)^{n-1},1)^T$ (as they cannot create K_3) whose first three entries are zeros, which is a contradiction.

Similarly, if $(1,1,0)^T$ is missing, one can deduce that, up to a row permutation, $M([3],)$ contains 7 distinct columns plus the columns $(1,0,0)^T$ and $(0,1,0)^T$ and, again, the multiplicity of $(0,0,0)^T$ is only one, which is a contradiction as above, completing the proof of the theorem. $\square$

We do not have any non-trivial results concerning K_k, $k \geq 4$ except that a computer search [10] showed that $\text{sat}(5, K_4) = 22$ and $\text{sat}(6, K_4) \leq 24$. (We do not know if a K_4-saturated 6×24-matrix discovered by a partial search is minimal.)

Problem 3.4 *For which* $k \geq 4$, $\text{sat}(n, K_k) = O(1)$*?*

4 Forbidding Small Matrices

Here we will try to compute $\text{sat}(n, F)$ for forbidden matrices with at most 3 rows.

4.1 Forbidding 1-Row Matrices

For any given 1-row matrix F, we can determine $\text{sat}(n, F)$ for all but finitely many values of n:

Theorem 4.1 *Let* $F = ((0)^m, (1)^l) = [mT_1^0, lT_1^1]$ *with* $l \geq m$*. Then, for* $n \geq l-1$*,*

$$\mathrm{sat}(n,F) = \begin{cases} 1, & \textit{if } l = 1 \textit{ and } n \geq 1, \\ 2, & \textit{if } m = 0 \textit{ and } l = 2, \\ l+1, & \textit{if } m = 0 \textit{ and } l \geq 3, \\ l+m-1, & \textit{if } m \geq 1 \textit{ and } l \geq 2. \end{cases}$$

Proof. Assume that $l \geq 2$, as the case $l = 1$ is trivial.

Suppose that $m = 0$. The case $l = 2$ is trivial, so assume $l \geq 3$. An example of $M \in \mathrm{SAT}(n,F)$ can be built for $n \geq l-1$ by taking T_n^0 plus T_n^n plus $\chi_{[n]\setminus\{i\}}$, $i \in [l-2]$, and $\chi_{[l-2]}$. Clearly, each row of $M \supseteq T_n^0$ has exactly $l-1$ ones so M is F-saturated and the upper bound follows.

On the other hand, suppose on the contrary that some F-saturated matrix M has $n \geq l-1$ rows and $c \leq l$ columns. As $c < 2^n$ and M contains the all-0 column, $c = l$ and some row $M(i,)$ contains at $l-1$ ones. As we are not allowed multiple columns in M, some other row, say $M(j,)$, has at most $l-2$ ones. Then $\chi_{\{j\}}$ is not a column of M because its ith entry is zero but its addition does not create l ones in a row. This contradiction establishes the case $m = 0$.

For $m > 1$, let M consist of T_n^n plus $\chi_{\{i\}}$, $i \in [m-2]$, plus $\chi_{[n]\setminus\{i\}}$, $i \in [l-1]$ and $\chi_{[m-1,l-1]}$. Clearly, each row of M contains l ones and $m-1$ zeros so any new column (which must contain at least one 0) creates an F-submatrix and the upper bound follows. The lower bound is trivial. □

Remark 4.2 The case when $n \leq l-2$ in Theorem 4.1 seems messy so we do not investigate it here.

4.2 Forbidding 2-Row Matrices

Now let us consider some particular 2-row matrices.

Let $F = lT_2^2$, that is, F consists of the column $(1,1)^T$ taken l times. Trivially, for $l = 1$ or 2, $\mathrm{sat}(n, lT_2^2) = n + l$ with $T_n^{\leq 1}$ and $[T_n^{\leq 1}, T_n^n]$ being the only extremal matrices. For $l \geq 3$, we can only show the following lower bound which is almost sharp for $l = 3$ when we can build a $3T_2^2$-saturated $n \times (2n+2)$-matrix by taking $T_n^{\leq 1}$, $\chi_{[n-1]}$, $\chi_{[n]}$, plus $\chi_{\{i,n\}}$ for $i \in [n-1]$.

Lemma 4.3 *For* $l \geq 3$ *and* $n \geq 3$*,* $\mathrm{sat}(n, lT_2^2) \geq 2n+1$*.*

Proof. Let $M = [T_n^{\leq 1}, M']$ be lK_2^2-saturated. Note that M' must have the property that every column χ_A with $A \in [n]^{(2)}$ either belongs to M' or creates an F-submatrix; in the latter case, exactly $l-1$ columns of M' have ones in both positions of A. Therefore, adding to M' some columns of T_n^2 (possibly multiple), we can obtain a new matrix M'' such that, for every $A \in [n]^{(2)}$, $M''(A,)$ contains the column $(1,1)^T$ exactly $l-1$ times. If we let set X_i be encoded by the ith row of M'' as its characteristic vector, we have that $|X_i \cap X_j| = l - 1$ for every $1 \leq i < j \leq n$. The result of Bose [8] (see [16, Theorem 14.6]), which can be viewed as an extension of the famous Fisher's inequality [12], asserts that the rows of M'' are linearly independent over the reals or M'' has two equal rows, say $X_i = X_j$. The last case is impossible because then $|X_i| = l - 1$ and each other X_h contains X_i as a subset; this in turn implies that the column $((1)^n)^T$ appears at least $l - 1 \geq 2$ times in M'' and (since $n \geq 3$) the same number of times in M', a contradiction. Thus the rank of M'' over the reals is n. Since every column added to M' when we were constructing M'' was already present in M', the matrices M' and M'' has the same rank over the reals. Thus M' has at least n columns and the lemma follows. $\square$

Let us show that Lemma 4.3 is sharp for some n. Suppose there exists a *symmetric* $(n, k, 2)$*-design* (meaning we have n k-sets $X_1, \ldots, X_n \in [n]^{(k)}$ such that every pair $\{i, j\} \in [n]^{(2)}$ is covered by exactly two X_i's). Let M be the $n \times n$-matrix whose rows are the characteristic vectors of the sets X_i. Then $[T_n^{\leq 1}, M]$ is a $3T_2^2$-saturated $n \times (2n + 1)$-matrix. For $n = 4$, we can take all 3-subsets of $[n]$. For $n = 7$, we can take the family $\{[7] \setminus Y_i : i \in [7]\}$, where $Y_1, \ldots, Y_7 \in [7]^{(3)}$ form the Fano plane. Constructions of such designs for $n = 16$, 37, 56, and 79 can be found in [9, Table 6.47].

Of course, the non-existence of a symmetric $(n, k, 2)$ design does not imply anything about $\mathrm{sat}(n, 3T_2^2)$, since a minimum $3T_2^2$-saturated matrix $[T_n^{\leq 1}, M]$ need not have the same number of ones in the rows of M.

Lemma 4.3 is not always optimal for $l = 3$. One trivial example is $n = 3$. Another one is $n = 5$:

Lemma 4.4 $\mathrm{sat}(5, 3T_2^2) = 12$.

Proof. Suppose on the contrary that we have a $3T_2^2$-saturated 5×11-matrix $[T_5^{\leq 1}, M]$. Let $X_1, \ldots, X_5$ be the subsets of $[5]$ encoded by the rows of M. Then $|X_i \cap X_j| = 2$ for $1 \leq i < j \leq 5$. If $X_i \subseteq X_j$ for distinct $i, j \in [5]$, then $|X_i| = 2$, every other X_h contains X_i as a subset, and M has two equal columns, a contradiction. In particular, $3 \leq |X_i| \leq 4$ for every $i \in [5]$. A simple case analysis gives a contradiction by assuming that each $|X_i| = 3$. Finally, if some $|X_i| = 4$, say $X_1 = [4]$, then each of $X_2, \ldots, X_5$ contains 5 and some two elements of $[4]$, and we can easily derive a contradiction. $\square$

Problem 4.5 *Determine* $\mathrm{sat}(n, 3T_2^2)$ *for every* n*.*

Remark 4.6 It is interesting to note that if we let $F = [lT_2^2, (0,1)^T]$ then the sat-function is bounded by a constant. Indeed, complete $M' = [\chi_{[n]\setminus\{i\}}]_{i \in [l]}$ to an arbitrary F-saturated matrix M. Clearly, in any added column all entries after the lth position are either zeros or ones; hence $\mathrm{sat}(n, F) \leq 2 \cdot 2^l$.

It is easy to compute $\mathrm{sat}(n, T_2^1)$ by observing that the n-row matrix M_Y whose columns encode $Y \subseteq 2^{[n]}$ is T_2^1-free if and only if Y is a chain, that is, of each two members of Y one is a subset of the other. Thus M_Y is T_2^1-saturated if and only if Y is a maximal chain without repeated entries. As all maximal chains in $2^{[n]}$ have size $n+1$, we conclude that

$$\mathrm{sat}(n, T_2^1) = \mathrm{forb}(n, T_2^1) = n+1, \quad n \geq 2.$$

Theorem 4.7 *Let* $F = [T_2^0, T_2^2] = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. *Then* $\mathrm{sat}(n, F) = 3$, $n \geq 2$.

Proof. For $n \geq 3$, the matrix M consisting of the columns $(0, 1, (1)^{n-2})^T$, $(1, 0, (1)^{n-2})^T$ and $(0, 0, (1)^{n-2})^T$ can be easily verified to be F-saturated and the upper bound follows.

Since $n = 2$ is trivial, let $n \geq 3$. Any 2-column F-free matrix $M \not\subseteq T_2^1$ is, without loss of generality, the following: we have rows $(0,0)$, $(1,1)$, $(1,0)$ and $(0,1)$ occurring, in this order, $n_{00} \geq 0$, $n_{11} \geq 1$, $n_{10} \leq 1$ and $n_{01} \leq 1$ times respectively. But then the addition of a new column $((0)^{n_{00}+1}, 1, 1, \dots)^T$ cannot create an F-submatrix. $\square$

Theorem 4.8 *Let* $F = T_2^{\geq 1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. *Then*

$$\mathrm{sat}(n, F) = \mathrm{forb}(n, F) = n+1, \quad n \geq 2.$$

Proof. Clearly, $\mathrm{forb}(n, F) \leq \mathrm{forb}(n, K_2) = n+1$.

Next, suppose that some $M \in \mathrm{SAT}(n, F)$ has two equal rows, for example, $M(1,) = M(2,) = ((1)^l, (0)^m)$. Let $X = [l]$ and $Y = [l+1, l+m]$. Define

$$A_i = \{j \in [l+m] : M(i,j) = 1\}, \quad i \in [n].$$

(For example, $A_1 = A_2 = X$.) As M is F-free, for every $i, j \in [n]$, the sets A_i and A_j are either disjoint or one is a subset of the other. For $i \in [3, n]$, let $b_i = 1$ if A_i strictly contains X or Y and let $b_i = 0$ otherwise (then A_i is contained in X or Y). Let $b_1 = 1$ and $b_2 = 0$.

Clearly, $C = (b_1, \dots, b_n)^T$ is not a column of M so its addition creates a forbidden submatrix, say $F \subseteq [M, C](\{i,j\},)$. Of course, $b_i = b_j = 0$ is impossible as $(0,0)^T \not\subseteq F$. If $b_i = b_j = 1$ then necessarily $A_i \cap A_j \neq \emptyset$ and $M(\{i,j\},) \supseteq (1,1)^T$ contains F. Finally, if $b_i \neq b_j$, *e.g.*, $b_i = 0$, $b_j = 1$ and $i < j$, then $A_i \supseteq A_j$ (as $(0,1)^T$ cannot be a column of $M(\{i,j\},)$), which implies $A_i = A_j$; but then we do not have a copy of F as $(1,0)^T$ is missing.

Thus, no F-saturated matrix M cannot have two equal rows and, by Theorem 2.1, $\mathrm{sat}(n, F) \geq n+1$ for any n. $\square$

Trivially, $\mathrm{sat}(n, [(0,1)^T, T_2^2]) = 2$ so we know the sat-function for any simple 2-row matrix.

4.3 Forbidding 3-Row Matrices

Here we consider some particular 3-row matrices. First we solve completely the case when $F = [T_3^0, T_3^3]$.

Theorem 4.9 *Let* $F = [T_3^0, T_3^3] = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$. *Then*

$$\operatorname{sat}(n, F) = \begin{cases} 7, & \text{if } n = 3 \text{ or } n \geq 6, \\ 10, & \text{if } n = 4 \text{ or } 5. \end{cases}$$

Proof. For the upper bound we define the following family of matrices.

$$M_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M_5 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$M_6 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

For any $n \geq 7$ let $M_n([6],) = M_6$ and $M_n(i,) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ for every $7 \leq i \leq n$. A computer search [10] showed that $\operatorname{sat}(n, F) = e(M_n)$ for $3 \leq n \leq 10$. It remains to show that

$$\operatorname{sat}(n, F) \geq 7 \tag{6}$$

for $n \geq 11$. In order to see this, we show the following result first.

Claim. If M is an F-saturated matrix of size $n \times m$ with $n \geq 11$ and $m \leq 6$ then M contains a row with all zero entries or with all one entries.

Suppose on the contrary that we have a counterexample M. We may assume that the first 6 entries of the first column of M are equal to 0. Consider a matrix $A = M([6], \{2, \ldots, m\})$. Note that every column of A contains at most two entries equal to 1, otherwise $M([6],) \supseteq F$. Hence, the number of 1's in A is at most $2(m-1)$.

By our assumption, each row of A has at least one entry 1. Since $2(m-1) < 12$, A has a row with precisely one entry equal to 1. We may assume that $A(1,1) = 1$ and $A(1,i) = 0$ for $2 \le i \le m$. Let C_2 be the second column of M (remember that $C_2(1) = A(1,1) = 1$).

Consider the n-column $C_3 = [0, C_2(\{2,\dots,n\})^T]^T$ which is obtained from C_2 by changing the first entry to 0. If it is not in M, then $F \subseteq [M, C_3]$. This copy of F has to contain the entry in which C_3 differs from C_2. But the only non-zero entry in Row 1 is $M(1,2)$; thus $F \subseteq [C_2, C_3]$, which is an obvious contradiction. Thus we may assume that C_3 is the third column of M.

We have to consider two cases. First, suppose that $C_2(\{2,\dots,n\})$ has at least one entry equal to 1. Without loss of generality, assume that $C_2(2) = C_3(2) = 1$.

It follows that $C_2(i) = C_3(i) = 0$ for $3 \le i \le 6$ (otherwise the first and the second columns of M would contain F). Let

$$B = M(\{3,4,5,6\}, \{4,\dots,m\}). \tag{7}$$

By our assumption, each row of B has at least one 1; in particular $m \ge 4$. Clearly, B contains at most $2(m-3) < 8$ ones. Thus, by permuting Rows $3,\dots,6$ and Columns $4,\dots,m$, we can assume that $B(1,1) = 1$ while $B(1,i) = 0$ for $2 \le i \le m-3$. Let C_4 be the fourth column of M and C_5 be such that C_4 and C_5 differ at the third position only, *i.e.*, $C_4(3) = 1$ and $C_5(3) = 0$. As before, C_5 must be in M, say it is the fifth column. Since $C_4(\{4,5,6\})$ has at most one 1, assume that $C_4(5) = C_4(6) = C_5(5) = C_5(6) = 0$. We need another column C_6 with $C_6(5) = C_6(6) = 1$ (otherwise the fifth or the sixth row of M would consist of all zero entries). In particular, $m = 6$. But now the new column C_7 which differs from C_6 at the fifth position only (*i.e.* $C_7(5) = 0$ and $C_7(i) = C_6(i)$ for $i \ne 5$) should be also in M, since M is F-saturated. This contradicts $e(M) = 6$. Thus the first case does not hold.

In the second case, we have $C_2(i) = C_3(i) = 0$ for every $2 \le i \le 6$. We may define B as in (7) and get a contradiction in the same way as above. This proves the claim.

Suppose on the contrary to the theorem that we can find an F-saturated matrix M with $n \ge 11$ rows and $m \le 6$ columns. By the claim, M has a constant row, say $M^T = [N^T, (0)^{(m)}]^T$. If C is an $(n-1)$-column missing from N, then the column $Q = (C^T, 1)^T$ is missing in M. Moreover, a copy of F in $[M, Q]$ cannot use the n-th row. Thus $F \subseteq [N, C]$, which means that $N \in \mathrm{SAT}(n-1, F)$ and $\mathrm{sat}(n-1, F) \le m \le 6$. Repeating this argument, we eventually conclude that $\mathrm{sat}(10, F) \le 6$, a contradiction to the results of our computer search. The theorem is proved. $\square$

Theorem 4.10 *Let* $F = [T_3^0, T_3^2, T_3^3] = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$. *Then*

$$\mathrm{sat}(n, F) = \begin{cases} 7, & \text{if } n = 3, 6 \text{ or } 7, \\ 9, & \text{if } n = 4 \text{ or } 5. \end{cases}$$

Moreover, for any $n \geq 8$, $\operatorname{sat}(n, F) \leq 7$.

Proof. For $n = 3, \ldots, 7$ the statement follows from a computer search [10] with the following F-saturated matrices.

$$M_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$M_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$M_6 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

For any $n \geq 7$ let $M_n([6],) = M_6$ and $M_n(i,) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$ for every $7 \leq i \leq n$ (*i.e.* the last row of M_6 is repeated $(n-6)$ times). It remains to show that M_n, $n \geq 8$, is F-saturated. Clearly, this is the case, since M_7 is F-saturated and F contains no pair of equal rows. $\square$

Conjecture 4.11 *Let $F = [T_3^0, T_3^2, T_3^3]$. Then $\operatorname{sat}(n, F) = 7$ for every $n \geq 8$.*

Theorem 4.12 *Let* $F = T_3^{\leq 2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$. *Then*

$$\operatorname{sat}(n, F) = \begin{cases} 7, & \text{if } n = 3, \\ 10, & \text{if } 4 \leq n \leq 6. \end{cases}$$

Moreover, for any $n \geq 7$, $\operatorname{sat}(n, F) \leq 10$.

Proof. For $n = 3, \ldots, 6$ the statement follows from a computer search [10] with the following F-saturated matrices.

$$M_4 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M_5 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

For any $n \geq 6$ let $M_n([5],) = M_5$ and $M_n(i,) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$ for every $6 \leq i \leq n$. It remains to show that M_n, $n \geq 7$, is F-saturated. Clearly, this is the case, since M_6 is F-saturated and F contains no pair of equal rows. $\square$

Conjecture 4.13 *Let* $F = T_3^{\leq 2}$. *Then* $\text{sat}(n, F) = 10$ *for every* $n \geq 7$.

Theorem 4.14 *Let* $F_1 = T_3^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, *and* $F_2 = [T_3^2, T_3^3] = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. *Then* $\text{sat}(n, F_1) = \text{sat}(n, F_2) = 3n - 2$ *for any* $3 \leq n \leq 6$. *Moreover, for any* $n \geq 7$, $\text{sat}(n, F_1) \leq 3n - 2$ *and* $\text{sat}(n, F_2) \leq 3n - 2$ *as well.*

Proof. Let $M_n = [T_n^0, T_n^1, T_n^n, \tilde{T}_n^2]$, where $\tilde{T}_n^2 \subseteq T_n^2$ and consists of all those columns of T_n^2 which have precisely one entry equals 1 either in the first or in the nth row, *e.g.*, for $n = 5$ we obtain

$$M_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Clearly, $e(M_n) = e(T_n^0) + e(T_n^1) + e(T_n^n) + e(\tilde{T}_n^2) = 1 + n + 1 + 2n - 4 = 3n - 2$. Moreover, since $\tilde{T}_n^2$ is F_1-admissible we get that M_n is both F_1 and F_2 admissible. Now we show that M_n is F_1-saturated (consequently, M_n is also F_2-saturated). Indeed, pick any column C which is not present in M_n. Such a column must contain at least 2 ones and 1 zero. Let $1 \leq i, j, k \leq n$ be the indices of C so that $c_i = 0$, $c_j = c_k = 1$. If $i = 1$ or $i = n$, then the matrix $M_n(\{i, j, k\},)$ contains F_1. Otherwise, $c_1 = c_n = 1$, and there also exists $1 < i < n$ such that $c_i = 0$. Here $M_n(\{1, i, n\},)$ contains F_1. Hence, $\text{sat}(n, F_1) \leq 3n - 2$ and $\text{sat}(n, F_2) \leq 3n - 2$ for any $n \geq 3$. A computer search [10] yields that these inequalities are equalities when $n = 3, \ldots, 6$. $\square$

Conjecture 4.15 *Let* $F_1 = T_3^2$ *and* $F_2 = [T_3^2, T_3^3]$. *Then* $\text{sat}(n, F_1) = \text{sat}(n, F_2) = 3n - 2$ *for every* $n \geq 7$.

Remark 4.16 It is not hard to see that $sat(n, F_1) \geq n + c\sqrt{n}$ for some absolute constant c and all $n \geq 3$. Indeed, let M be an $n \times (n + 2 + \lambda)$ F_1-saturated matrix of size $\text{sat}(n, F_1)$ for some $\lambda = \lambda(n)$. We may assume that $M(, [n+2]) = [T_n^0, T_n^1, T_n^n]$. Moreover, we assume that every column of matrix $M([\lambda], \{n+3, \ldots, n+2+\lambda\})$

contains at least one entry equal to 1 (there must be a permutation of the rows of M satisfying this requirement). We claim that all rows of $M(\{\lambda+1,\ldots,n\},\{n+3,\ldots,n+2+\lambda\})$ are different. Suppose not. Then, there are indices $\lambda+1 \le i,j \le n$ such that $M(i,\{n+3,\ldots,n+2+\lambda\}) = M(j,\{n+3,\ldots,n+2+\lambda\})$. Now consider a column C in which the only nonzero entries correspond to i and j. Clearly, C is not present in M, since the first λ entries of C equal 0. Moreover, since M is F_1-saturated, the matrix $[M,C]$ contains F_1. In other words, there are three rows in M which form F_1 as a submatrix. Note that the ith and jth row must be among them. But this is not possible since F_1 has no pair of equal rows.

Let $M_0 = M(\{\lambda+1,\ldots,n\},\{n+3,\ldots,n+2+\lambda\})^T$. Clearly, M_0 is F_1 admissible. Anstee and Sali showed (see Theorem 1.3 in [3]) that $\mathrm{forb}(\lambda,F_1) = O(\lambda^2)$. That means that $n-\lambda = O(\lambda^2)$, and consequently, $\lambda = \Omega(\sqrt{n})$. Hence, $\mathrm{sat}(n,F_1) = e(M) \ge n + \Omega(\sqrt{n})$, as required.

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