Planar Ramsey Numbers for Small Graphs

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Abstract

Given two graphs G_1 and G_2, the planar Ramsey number $PR(G_1, G_2)$ is the smallest integer n such that every planar graph on n vertices either contains a copy of G_1 or its complement contains a copy of G_2. So far, the planar Ramsey numbers have been determined, when both, G_1 and G_2 are complete graphs or both are cycles. By combining computer search with some theoretical results, in this paper we compute most of the planar Ramsey numbers $PR(G_1, G_2)$, where each of G_1 and G_2 is a complete graph, a cycle or a complete graph without one edge.

1 Introduction

Given two arbitrary graphs G_1 and G_2, *the planar Ramsey number* $PR(G_1, G_2)$ is the smallest integer n such that every planar graph F of order n either contains a copy of G_1 or its complement F^c contains a copy of G_2. *A planar Ramsey graph* (with respect to the pair (G_1, G_2)) is any planar graph F with $PR(G_1, G_2) - 1$ vertices and such that neither $F \supseteq G_1$, nor $F^c \supseteq G_2$.

In 1969, Walker [7] and, independently in 1993, Steinberg and Tovey [6] introduced planar Ramsey numbers and calculated them for all pairs of complete graphs (K_m, K_n) (Table 2).

Let C_n denote the cycle of length n. Bielak and Gorgol [2, 3] computed some special cases of $PR(C_4, K_n)$, and recently, Gorgol and Ruciński [5] determined planar Ramsey numbers for all pairs of cycles (C_m, C_n) (Table 3).

We use notation kG for a vertex disjoint union of k copies of a graph G, $G - v$ for the graph obtained from G by removing its vertex v, and, given $H \subset G$, $G - H$ for the graph obtained from

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G by removing all edges of one copy of H in G. One of the main characters in this paper, next to K_n and C_n, is the graph $K_n^- = K_n - K_2$.

We obtain precise values of most planar Ramsey numbers $PR(G_1, G_2)$, where G_1 and G_2 belong to the infinite family of graphs

$$\bigcup_n \{K_n, K_n^-, C_n\}.$$

In our proofs we combine computer search, described in the next section, with some theoretical results. Our strategy is first to try to set up an upper bound, either by a theoretical argument or by computer search, and then provide a planar Ramsey graph with respect to a given pair (G_1, G_2), establishing a matching lower bound.

Similarly to the classical Ramsey numbers (where we do not assume that F is planar), the planar Ramsey numbers are monotone in the sense that for all $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ we have $PR(H_1, H_2) \leq PR(G_1, G_2)$.

However, in contrast to Ramsey numbers, the planar Ramsey numbers are not symmetric, i.e., typically $PR(G_1, G_2)$ is not equal to $PR(G_2, G_1)$. For instance, if G_1 is non-planar, then $PR(G_1, G_2)$ is determined solely by G_2, and is equal to the smallest integer n such that the complement of every planar graph F contains a copy of G_2. Thus, for all non-planar G_1' and G_1'' and all G_2 we have

$$PR(G_1', G_2) = PR(G_1'', G_2). \tag{1}$$

Our results are summarized in Tables 2-10. We begin with those cases where all numbers can be computed, which happens when $G_2 = C_n$. Unfortunately, in the remaining cases there are still infinite sequences of pairs of graphs for which the planar Ramsey number is not known. In such cases, whenever available, we provide best known lower and/or upper bounds. If we do not know any reasonable bounds, we mark the corresponding entries with *'?'*.

Some generalizations, open problems and conjectures are contained in Section 6.

2 Facts based on computer search

In this section, we present results obtained by computer search. Our technique is based on the program *plantri* [8] developed by G. Brinkmann and B. McKay, which generates certain types of planar graphs. The junior author wrote a program [9], which for given 2-connected graphs G_1 and G_2, checks whether a planar graph generated by *plantri* contains a copy of G_1 or its complement contains a copy of G_2. Due to time complexity, we were able to search only graphs of very small order. Fortunately, quite often the most difficult part of determining a sequence of planar Ramsey numbers lies in small cases.

The results obtained by computer search are presented in Table 1. Each row corresponds to one instance of computer search. The first column shows the number $|V(F)|$ of vertices in the graphs F under search. In the second column we describe the structure of the planar graphs F being searched (either G_1-free for some G_1 or maximal), while in the third one we list subgraphs which

	$\|V(F)\|$	Structure of F	Subgraphs of F^c	Conclusion
1	6	K_3-free	K_4^-	$PR(K_3, K_4^-) \leq 6$
2	7	K_4^--free	C_3, C_4, C_5	$PR(K_4^-, C_n) \leq 7$, for $n = 3, 4, 5$
3	7	C_4-free	K_4^-	$PR(C_4, K_4^-) \leq 7$
4	8	maximal	C_4	$PR(G, C_4) \leq 8$
5	8	K_4-free	C_5	$PR(K_4, C_5) \leq 8$
6	8	K_4^--free	C_6	$PR(K_4^-, C_6) \leq 8$
7	9	maximal	C_3, C_5, C_6, C_7	$PR(G, C_n) \leq 9$, $n = 3, 5, 6, 7$
8	9	K_3-free	K_5^-	$PR(K_3, K_5^-) \leq 9$
9	9	K_4^--free	K_4^-	$PR(K_4^-, K_4^-) \leq 9$
10	9	C_5-free	K_4^-	$PR(C_5, K_4^-) \leq 9$
11	10	maximal	C_8	$PR(G, C_8) \leq 10$
12	10	K_4^--free	K_4	$PR(K_4^-, K_4) \leq 10$
13	11	maximal	C_9	$PR(G, C_9) \leq 11$
14	11	C_4-free	K_5^-	$PR(C_4, K_5^-) \leq 11$
15	12	maximal	C_{10}	$PR(G, C_{10}) \leq 12$
16	12	K_3-free	K_6^-	$PR(K_3, K_6^-) \leq 12$
17	13	maximal	C_{11}	$PR(G, C_{11}) \leq 13$
18	13	K_4^--free	K_5^-	$PR(K_4^-, K_5^-) \leq 13$
19	13	C_5-free	K_5^-	$PR(C_5, K_5^-) \leq 13$
20	14	K_4^--free	K_5	$PR(K_4^-, K_5) \leq 14$
21	14	C_4-free	K_6^-	$PR(C_4, K_6^-) \leq 14$
22	15	K_3-free	K_7^-	$PR(K_3, K_7^-) \leq 15$
23	17	C_5-free	K_6^-	$PR(C_5, K_6^-) \leq 17$

Table 1: Facts based on computer search.

were confirmed to be present in the complement F^c of each graph. The last column states an upper bound on the respective planar Ramsey number resulting from the search.

For instance, the first row of Table 1 describes the search for a copy of K_4^- in the complements of all 6-vertex triangle-free planar graphs. As a copy of K_4^- was found in each of them, the conclusion is that $PR(K_3, K_4^-) \leq 6$. In fact, we had to check every connected 6-vertex planar graph to determine if it is triangle-free, and only then search its complement for a copy of K_4^-.

In all cases considered we could have restricted our search to connected graphs only (which can be performed in *plantri*), making the number of graphs to be searched significantly smaller. This was possible, because G_1 is always 2-connected, and so, joining two components by a single edge cannot introduce a copy of G_1 to F (and, of course, cannot introduce a copy of G_2 to F^c.)

In a couple of cases, we searched only the maximal planar graphs, that is, triangulations (again, there is such an option in *plantri*) with a given number of vertices, and the obtained bounds on $PR(G_1, G_2)$ are then valid for all graphs G_1.

3 Cycles in the complements

In this section we present results that determine all planar Ramsey numbers $PR(G_1, G_2)$, where $G_1 = K_m$ or $G_1 = K_m^-$, while $G_2 = C_n$, $m, n \geq 3$.

We first state a result which illustrates very well how a theoretical proof can be simplified by using computer search on small graphs. The original proof was much longer and complicated, because certain facts from Table 1 were not available at that time (cf. Theorem 4 in [5]).

Lemma 3.1 ([5]) *The complement of every planar graph F on $t \geq 9$ vertices contains C_{t-2}.*

Proof. The proof is by induction on t. For $t = 9, ..., 13$ the statement follows by Table 1, rows 7, 11, 13, 15, 17. If $t \geq 14$, then let v be a vertex of degree at most 5 in F (such a vertex always exists in a planar graph). To the graph $F - v$ apply the induction assumption. Thus, there is a copy C of the cycle C_{t-3} in $(F - v)^c$. Note that v has at least $t - 1 - 5 - 2 = t - 8$ neighbors on C, and for $t \geq 14$ the inequality $t - 8 > (t - 3)/2$ holds. This means that two of these neighbors are consecutive vertices of C, and, consequently, C can be extended to a cycle of length $t - 2$ in F^c.

□

Lemma 3.1 is not true for $t = 8$ (see Figure 1 in [5]).

Now we are ready to determine all planar Ramsey numbers $PR(K_m, C_n)$. Note that by (1) we have $PR(K_m, C_n) = PR(K_5, C_n)$, for all $m \geq 6$ and all n.

Theorem 3.2 *The planar Ramsey numbers $PR(K_m, C_n)$ are as shown in Table 4.*

Proof. Note that the first column and the first row contain the values of $PR(K_m, K_3)$ and $PR(C_3, C_n)$ which have been determined already in [6] and [5], respectively (see Tables 2 and 3).

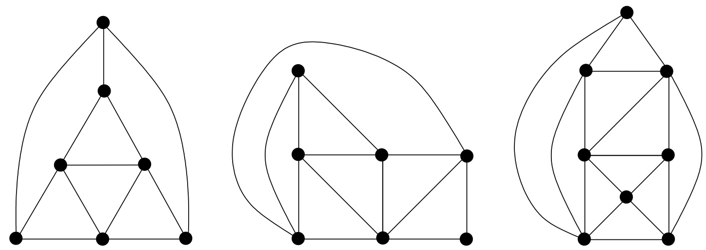
Next, we determine the number $PR(K_4, C_5)$. The upper bound is given by Table 1, row 5. To establish the corresponding lower bound, observe that the graph in Figure 1(b) is a planar Ramsey graph for (K_4, C_5).

The upper bound $PR(K_m, C_4) \leq 8$ for $m = 4, 5$ follows from Table 1, row 4, while the bounds $PR(K_5, C_5) \leq 9$ and $PR(K_m, C_6) \leq 9$ for $m = 4, 5$ follow from Table 1, row 7. The corresponding planar Ramsey graphs are, respectively, the graph in Figure 1(a), $2K_4$ and the graph in Figure 1(c).

Finally, the upper bound $PR(K_m, C_n) \leq n + 2$ for $m = 4, 5$ and $n \geq 7$ follows from Lemma 3.1, while the corresponding planar Ramsey graph is the complete bipartite graph $K_{2,n-1}$.

□

Note that Table 4 differs from Table 3 only in the second row ($m = 4$) and in the two entries with $m = 5$ and $n = 4, 6$. In all other cases the above proof could be shortcut by using the monotonicity of



(a) (b) (c)

Figure 1: K_4-free planar graphs on 7 [(a),(b)] and 8 [(c)] vertices with, respectively, no C_4, C_5 and C_6 in their complements.

planar Ramsey numbers, that is, the inequality $PR(K_m, C_n) \geq PR(C_m, C_n)$, instead of constructing explicit planar Ramsey numbers.

Now we determine all numbers $PR(K_m^-, C_n)$.

Theorem 3.3 *The planar Ramsey numbers $PR(K_m^-, C_n)$ are as shown in Table 5.*

Proof. Note that by (1), for all $m \geq 7$ and all n, we have

$$PR(K_m^-, C_n) = PR(K_6^-, C_n) = PR(K_5, C_n).$$

In particular, the case $m = 6$ follows.

By monotonicity, for all $m, n \geq 3$ we have

$$PR(K_{m-1}, C_n) \leq PR(K_m^-, C_n) \leq PR(K_m, C_n). \tag{2}$$

Thus, for $m = 4$ and $n \geq 7$, as well as for $m = 5$ and $n \geq 6$, Theorem 3.3 follows from Theorem 3.2. This is because in Tables 4 and 5, not only all these entries coincide, but each of them has in Table 4 the same value directly above in the previous row. Consequently, both estimates in (2) are equal.

Turning to the case $m = 3$, it is easy to see that $PR(K_3^-, C_3) = 5$ (the graph $2K_2$ is the only planar Ramsey graph in this case), and that, by Dirac's Theorem (see [4], $PR(K_3^-, C_n) = n$ for $n \geq 4$.

The upper bounds on the numbers $PR(K_4^-, C_n)$, $n = 3, 4, 5, 6$, are given in Table 1, rows 2 and 6. The lower bound $PR(K_4^-, C_3) \geq 7$ follows by considering the graph $2K_3$, while the respective lower bounds on $PR(K_4^-, C_n)$, $n = 3, 4, 5$, follow from the left-hand side of (2) and Theorem 3.2.

Further, we have $PR(K_5^-, C_5) \leq 9$ by Table 1, row 7 , and $PR(K_5^-, C_5) \geq 9$ by considering $2K_4$.

Finally, by Theorem 3.2, the left-hand side of (2), and by Table 1, row 7 again,

$$9 = PR(K_4, C_3) \leq PR(K_5^-, C_3) \leq 9,$$

and, similarly, using Table 1, row 4 instead,

$$8 = PR(K_4, C_4) \leq PR(K_5^-, C_4) \leq 8.$$

□

4 Almost complete graphs

In this section, we compute almost all instances of the numbers $PR(G_1, G_2)$, where each of G_1 and G_2 is either a complete graph, or a complete graph without one edge. The most challenging was the case of $PR(K_m, K_n^-)$, and we consider Theorem 4.3 below as the main result of this paper.

We begin with a useful fact which generalizes the formula $PR(K_m, K_n) = 4n-3$ for $m \geq 5$. The proof is very simple, because it relies on the celebrated Four Color Theorem.

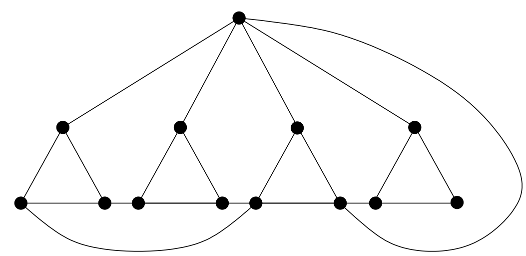


Figure 2: A K_4^- - free planar graph on 13 vertices with no K_5 in its complement.

Proposition 4.1 *Let G be a connected graph with at least 5 vertices. Then $PR(G, K_n) = 4n - 3$ for every $n \geq 2$.*

Proof. By the Four Color Theorem, in every planar graph on $4n - 3$ vertices there exists an independent set of size at least $\lceil (4n-3)/4 \rceil = n$. Thus, $PR(G, K_n) \leq 4n - 3$ for any graph G.

To prove the lower bound, notice that the graph $F = (n-1)K_4$ is a planar Ramsey graph with respect to the pair (G, K_n) for any G such that $F \not\supseteq G$.

□

Theorem 4.2 *The planar Ramsey numbers $PR(K_m^-, K_n)$ are as shown in Table 6.*

Proof. For $m = 5, 6$ the result follows from Proposition 4.1. For $m = 4$ and $n = 3, 4, 5$ the upper bounds are provided by Table 1, rows 2, 12, 20, while the lower bounds follow by considering $2K_3$, $3K_3$ and the graph in Figure 2, respectively. In addition, the lower bound $PR(K_4^-, K_6) \geq 17$ follows by considering the graph in Figure 3.

□

Theorem 4.3 *The planar Ramsey numbers $PR(K_m, K_n^-)$ are as shown in Table 7.*

Proof. The case $n = 3$ can be verified by hand. Indeed, if there is no K_3^- in F^c, then F is of the form $K_t - sK_2$, $s \leq t/2$. Since $K_5 - 2K_2 \supset K_3$ and $K_7 - 3K_2$ in non-planar, we get the bopunds $PR(K_3, K_3^-) \leq 5$ and $PR(K_m, K_3^-) \leq 7$ for all $m \geq 4$. The corresponding lower bounds are provided by the planar Ramsey graphs C_4 and $K_6 - 3K_2$, respectively.

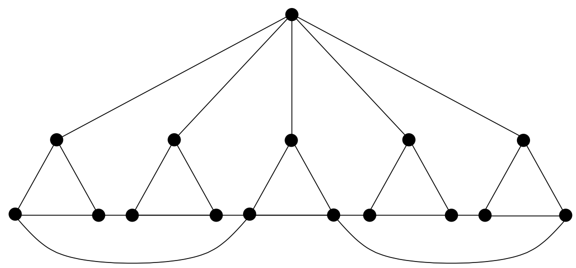


Figure 3: A K_4^- - free (in fact C_4 - free) planar graph on 16 vertices with no K_6 in its complement.

The case $m = 3$ is much harder. The lower bound comes from monotonicity. Indeed, for all $n \geq 4$, we have

$$PR(K_3, K_n^-) \geq PR(K_3, K_{n-1}) = 3n - 6.$$

The upper bounds for $n = 4, 5, 6, 7$ are provided by Table 1, rows 1, 8, 16, 22.

The following two facts complete the proof of Theorem 4.3 in the remaining cases of $m = 4, 5$.

Fact 4.4 *For all $n \geq 4$, the complement of every planar graph on $4n - 5$ vertices contain a copy of K_n^-.*

Fact 4.5 $PR(K_5, K_n^-) \geq PR(K_4, K_n^-) \geq 4n - 5$ *for every* $n \geq 4$.

Proof of Fact 4.4. Let F be a planar graph on $4n-5$ vertices. If $\alpha(F) \geq n$, then $F^c \supseteq K_n \supset K_n^-$. Otherwise, by the Four Color Theorem, the vertex set $V(F)$ can be partitioned into four independent sets V_i, $i = 1, 2, 3, 4$, such that $|V_1| = |V_2| = |V_3| = n - 1$ and $|V_4| = n - 2$.

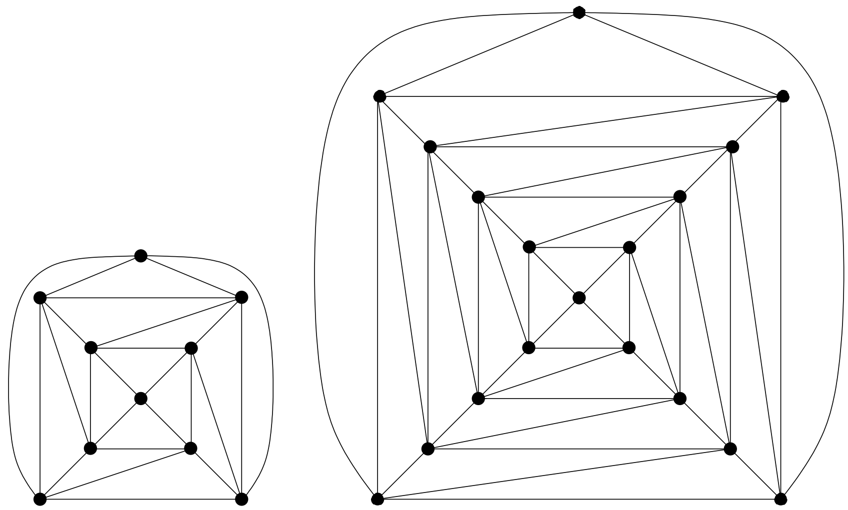
If for some $i = 1, 2, 3$ there exists a vertex $v \notin V_i$ having precisely one neighbor in V_i, then the subgraph $F^c[V_i \cup \{v\}]$ is isomorphic to K_n^-. Suppose that each vertex $v \notin V_i$ has at least 2 neighbors in V_i, $i = 1, 2, 3$. Then the number of edges between V_i and V_j for $1 \leq i < j \leq 3$ is at most $2(n-1)$ and the number of edges between V_4 and V_i, $i = 1, 2, 3$, is at most $2(n-2)$. Hence, altogether $|E(F)| \geq 12n - 18$. On the other hand, as F is planar, $|E(F)| \leq 3(4n-5) - 6 = 12n - 21 < 12n - 18$ – a contradiction.

□

Proof of Fact 4.5. The following construction of a sequence of planar graphs F_t, $t \geq 4$, such that $V(F_t) = 4t - 6$, $F_t \not\supseteq K_4$ and $F_t^c \not\supseteq K_t^-$ was inspired by computer search.

The graph F_t consists of a sequence of $t-2$ copies of the cycle C_4, denoted by $C^1, \ldots, C^{t-2}$, such that C^{i+1} surrounds C^i, $i = 2, \ldots, t$, and two extra vertices, v and w. The vertex v is connected to all vertices of C^1 and w is connected to all vertices of C^{t-2}. In addition, denoting the vertices of C^i by u_j^i, $j = 1, 2, 3, 4$, each u_j^i is connected to u_j^{i+1} and u_{j+1}^{i+1}, $i = 1, \ldots, t-3$, $j = 1, 2, 3, 4$ (the addition in the subscript is modulo 4). The graphs F_4 and F_6 are shown in Figure 4.

Since the neighborhood of no vertex contains a triangle, $F_t \not\supseteq K_4$. We will show by induction that for every $t \geq 4$ we have $F_t^c \not\supseteq K_t^-$. It is easy to check that for $t = 4$ the statement is true.



(a) (b)

Figure 4: (a) A K_4-free planar graph on 10 vertices with no K_4^- in its complement; (b) a K_4-free planar graph 14 vertices and K_5^- in its complement.

Suppose that this is true for F_{t-1}, but F_t^c contains an induced subgraph H isomorphic to K_t^- or K_t.

Note that for every $i = 1, \ldots, t-2$, we have $|V(H) \cap V(C^i)| \leq 2$. If $V(H) \cap V(C^1) = \emptyset$, then by removing C^1 and reconnecting v with all vertices of C^2, we obtain a copy of K_{t-1}^- in F_{t-1} – a contradiction. Hence, $V(H) \cap V(C^1)| \neq \emptyset$, and, by symmetry, $V(H) \cap V(C^{t-2}) \neq \emptyset$.

Call a vertex of H *complete* if its degree in H is $t-1$. Since only at most two vertices of H are not complete, and, if exist, they are connected by an edge, either $V(H) \cap V(C^1)$ or $V(H) \cap V(C^{t-2})$ consists of complete vertices only. Without loss of generality assume that it is $V(H) \cap V(C^1)$. Consequently, $v \notin V(H)$.

If $|V(H) \cap V(C^1)| = 1$, then, as before, we obtain a copy of K_{t-1}^- in F_{t-1} – a contradiction. If $|V(H) \cap V(C^1)| = 2$, then $V(H) \cap V(C^2) = \emptyset$. Let $u \in V(H) \cap V(C^1)$. By removing from F_t all vertices of the set $V(C^1) \cup \{v\} \setminus \{u\}$ and connecting u with all vertices of C^2, we again obtain a copy of K_{t-1}^- in F_{t-1} – a contradiction.

□

Theorem 4.6 *The planar Ramsey numbers $PR(K_m^-, K_n^-)$ are as shown in Table 8.*

Proof. For $m = 5, 6$ and for all $n \geq 4$, by Fact 4.5 and Theorem 4.3 we have

$$4n - 5 \leq PR(K_5^-, K_n^-) \leq PR(K_6^-, K_n^-) \leq PR(K_6, K_n^-) = 4n - 5.$$

The equations $PR(K_4^-, K_4^-) = 9$ and $PR(K_4^-, K_5^-) = 13$ follow from Table 1, rows 9, 18 (upper bounds), and by considering the graphs in Figure 5 (lower bounds).

□

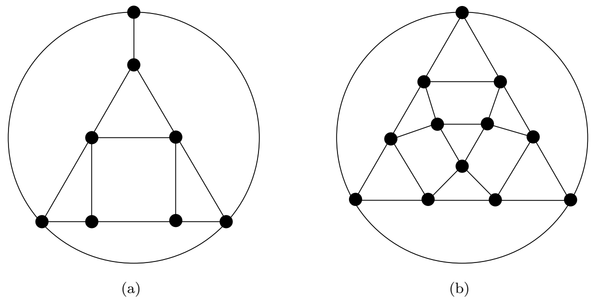


Figure 5: (a) A K_4^- - free planar graph on 8 vertices with no K_4^- in its complement; (b) A K_4^- - free planar graph on 12 vertices with no K_5^- in its complement.

5 Independent sets in planar graphs with forbidden cycles

In this section we study the Ramsey numbers $PR(G_1, G_2)$ where G_1 is a cycle and G_2 is a complete or almost complete graph. These numbers turned out to be the hardest to compute. This is because planar graphs with forbidden cycles have a more complex structure than those with forbidden cliques.

Theorem 5.1 *The planar Ramsey numbers $PR(C_m, K_n)$ are as shown in Table 9. In particular, for $m = 4$ and $n \geq 7$, we have*

$$3n + \lfloor (n-1)/5 \rfloor - 2 \leq PR(C_4, K_n) \leq \min\{29 + 5(n-9), 4n - 3\}. \quad (3)$$

Proof. For $m = 3$, by [6, 7], we have $PR(C_3, K_n) = PR(K_3, K_n) = 3n - 6$. For $m = 4$ and $n = 3, 4, 5, 6$ the planar Ramsey numbers were calculated by Bielak and Gorgol [2, 3].

The lower bound on $PR(C_4, K_n)$ for $n \geq 7$ is provided by a construction from [2], the special case of which is the 16-vertex graph in Figure 3. It does not contain C_4 and there is no K_6 in the complement. For arbitrary $n \geq 7$ take $\lfloor (n-1)/5 \rfloor$ disjoint copies of this graph and $n-1-5\lfloor (n-1)/5 \rfloor$ copies of K_3. This graph still does not contain C_4, while the largest clique in the complement is of order at most $n - 1$. Hence, the left hand side of (3) holds. In particular, $PR(C_4, K_7) \geq 20$ and $PR(C_4, K_8) \geq 23$.

For the upper bound on $PR(C_4, K_n)$, we need the following simple fact, the standard proof of which is omitted.

Fact 5.2 *Every C_4-free, t-vertex planar graph has at most $15(t-2)/7$ edges. In particular, every C_4-free, t-vertex, $t \leq 29$ planar graph has a vertex of degree at most 3.*

Take an arbitrary, C_4 - free planar graph on 21 vertices, and let v be a vertex of degree at most 3. If we remove this vertex and all its neighbors, the obtained graph has at least 17 vertices, and, since $PR(C_4, K_6) \leq 17$, contains K_6 in its complement. This clique forms with v a clique K_7, and so $PR(C_4, K_7) \leq 21$. In a similar fashion, we obtain bounds $PR(C_4, K_8) \leq 25$ and $PR(C_4, K_9) \leq 29$, and, for $n \geq 10$, the RHS of (3), where the second term under the minimum comes from Table 2 and takes over for $n \geq 13$. Finally, for $m \geq 5$ the result follows from Proposition 4.1.

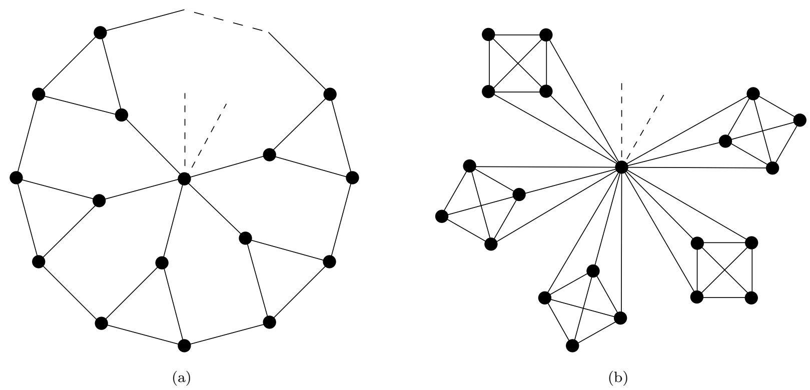


Figure 6: (a) A C_4 - free planar graph on $3(n-2)+1$ vertices, $n \geq 5$, with no K_n^- in its complement (the vertex in the middle has degree $n-2$); (b) a C_m - free planar graph on $4(n-2)+1$ vertices, $m \geq 6$, $n \geq 4$ with no K_n^- in its complement (the vertex in the middle has degree $3(n-2)$).

□

Theorem 5.3 *The planar Ramsey numbers $PR(C_m, K_n^-)$ are as shown in Table 10.*

Proof.

The case $m = 3$ follows from Theorem 4.3 (see Table 7). For the lower bounds on $PR(C_m, K_3^-)$, consider C_4 for $m = 3$ and $m = 5$, K_3 for $m = 4$, $K_5 - K_2$ for $m = 6$ and $K_6 - 3K_2$ for $m \geq 7$. For the upper bounds, suppose that F is a planar graph whose complement does not contain K_3^-. Then F^c is a union of disjoint edges and isolated vertices. Hence, we may assume that F^c has at most one isolated vertex (none if $|V(F)|$ is even).

It can be verified by hand that $K_5 - 2K_2 \supset C_3$ as well as $K_4 - 2K_2 = C_4$. In fact, for all $m = 4, 5, 6$, the graph $K_m - \lfloor m/2 \rfloor K_2$ is hamiltonian. For $m \geq 7$, the graph $K_7 - 3K_2$ is non-planar, so F^c must contain K_3^-.

The lower bounds on the numbers $PR(C_4, K_4^-)$, $PR(C_4, K_n^-)$, $n = 5.6, 7$, $PR(C_5, K_4^-)$, $PR(C_5, K_5^-)$, and $PR(C_5, K_6^-)$ are provided by the planar Ramsey graphs $2K_3$, the graph in Figure 6(a) with $n = 5$, $n = 6$ and $n = 7$, $2K_4$, $3K_4$ and $4K_4$, respectively. The corresponding upper bounds (except for $PR(C_4, K_7^-)$ for which we do not have any upper bound) are given in Table 1, rows 3, 14, 21, 10, 19, 23.

By monotonicity, $PR(C_4, K_n^-) \geq PR(C_4, K_{n-1})$. Thus, by the LHS of (3), for $n \geq 8$ we have

$$PR(C_4, K_n^-) \geq 3n + \lfloor (n-2)/5 \rfloor - 5 \tag{4}$$

Note that for $n = 7$, the RHS of (4) equals 17 and coincides with the lower bound on $PR(C_4, K_7^-)$ obtained above via Figure 6(a). However, for $n > 7$, the lower bound of $3n - 4$ resulting from the graph in Figuree 6(a) is worse than (4).

The lower bounds on the numbers $PR(C_5, K_n^-) \geq 4n - 7$ for $n \geq 7$, and $PR(C_m, K_n^-) \geq 4n - 6$ for $m \geq 6$ and $n \geq 4$ follow from considering the planar Ramsey graphs $(n-2)K_4$ and the graph in Figure 6(b).

We leave it to the reader to check that the graphs in Figure 6 satisfy the properties stated in the caption.

□

6 Concluding remarks

Some theorems presented in the previous sections can be formulated more generally.

We have remarked in the Introduction that if G_1 is nonplanar then $PR(G_1, G_2)$ does not depend on G_1. But even when G_1 is planar, the number $PR(G_1, G_2)$ might still be solely determined by G_2, in the sense that the complement of every planar graph of order at least $PR(G_1, G_2)$ contains G_2. Then, in fact, we have

$$PR(G, G_2) = PR(G_1, G_2) \tag{5}$$

for all graphs G such that G is not contained in some planar Ramsey graph F with respect to G_1 and G_2. Indeed, since $F \not\supseteq G$ and $F^c \not\supseteq G_2$, $PR(G, G_2) \geq |V(F)| + 1 = PR(G_1, G_2)$. On the other hand, we know that the complement of every planar graph of order at least $PR(G_1, G_2)$ contains G_2. Thus $PR(G, G_2) \leq PR(G_1, G_2)$.

As an application, we immediately determine the following planar Ramsey numbers.

Corollary 6.1 *(a) $PR(G, C_3) = PR(G, C_5) = PR(G, C_7) = 9$ for every $G \not\subseteq 2K_4$.*

(b) $PR(G, C_n) = n + 2$ for every $n \geq 7$ and $G \not\subseteq K_{2,n-1}$.

(c) $PR(G, K_n^-) = 4n - 5$ for every $n \geq 4$ and $G \not\subseteq F_n$, where F_n is the graph from the proof of Theorem 4.3.

Proof. Part (a) follows from Table 1, part (b) – from Lemma 3.1 and part (c) from Fact 4.4.

□

We conclude this section with a summary of what we do not know about the planar Ramsey numbers studied in this paper. The following sequences of planar Ramsey numbers have not been determined so far.

- $PR(K_4^-, K_n)$ for $n \geq 6$
- $PR(K_3, K_n^-) = PR(C_3, K_n^-)$ for $n \geq 8$
- $PR(K_4^-, K_n^-)$ for $n \geq 6$
- $PR(C_4, K_n)$ for $n \geq 7$
- $PR(C_m, K_n^-)$ for $m = 4, 5$ and $n \geq 7$
- $PR(C_m, K_n^-)$ for $m \geq 6$ and $n \geq 4$

Based on computer search (and our instinct) we dare to state the following conjectures.

Conjecture 6.2 *(a)* $PR(K_3, K_n^-) = PR(C_3, K_n^-) = 3n - 6$ *for* $n \geq 5$ *(true for* $n = 5, 6, 7$*)*

(b) $PR(C_5, K_n^-) = 4n - 7$ *for* $n \geq 3$ *(true for* $n = 3, 4, 5, 6$*)*

(c) $PR(C_m, K_n^-) = 4n - 6$ *for* $m \geq 6$ *and* $n \geq 4$

7 The tables

<table>
<tr><th>m \ n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>...</th></tr>
<tr><td>3</td><td colspan="7">$3n-3$</td></tr>
<tr><td>4</td><td colspan="7" rowspan="2">$4n-3$</td></tr>
<tr><td>≥ 5</td></tr>
</table>

Table 2: *(Walker [7], Steinberg, Tovey [6])* The planar Ramsey numbers $PR(K_m, K_n)$.

<table>
<tr><th>m \ n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>...</th></tr>
<tr><td>3</td><td>6</td><td>7</td><td colspan="6">$n+2$</td></tr>
<tr><td>4</td><td>7</td><td>6</td><td>7</td><td colspan="5">$n+1$</td></tr>
<tr><td>5</td><td rowspan="4">9</td><td rowspan="2">7</td><td rowspan="4">9</td><td rowspan="2">8</td><td colspan="4" rowspan="4">$n+2$</td></tr>
<tr><td>6</td></tr>
<tr><td>7</td><td rowspan="2">8</td><td rowspan="2"></td></tr>
<tr><td>⋮</td></tr>
</table>

Table 3: *(Gorgol, Ruciński [5])* The planar Ramsey numbers $PR(C_m, C_n)$.

<table>
<tr><th>m \ n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>...</th></tr>
<tr><td>3</td><td>6</td><td>7</td><td colspan="2"></td><td colspan="4" rowspan="3">$n+2$</td></tr>
<tr><td>4</td><td rowspan="2">9</td><td rowspan="2">8</td><td></td><td rowspan="2">9</td></tr>
<tr><td>≥ 5</td><td></td></tr>
</table>

Table 4: The planar Ramsey numbers $PR(K_m, C_n)$.

<table>
<tr><th>m \ n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>...</th></tr>
<tr><td>3</td><td>5</td><td colspan="7">n</td></tr>
<tr><td>4</td><td colspan="2">7</td><td colspan="2"></td><td colspan="4" rowspan="3">$n+2$</td></tr>
<tr><td>5</td><td rowspan="2">9</td><td rowspan="2">8</td><td colspan="2" rowspan="2">9</td></tr>
<tr><td>≥ 6</td></tr>
</table>

Table 5: The planar Ramsey numbers $PR(K_m^-, C_n)$.

<table>
<tr><th>m \ n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>...</th></tr>
<tr><td>3</td><td colspan="8">$2n-1$</td></tr>
<tr><td>4</td><td>7</td><td>10</td><td>14</td><td>≥ 17</td><td colspan="4">?</td></tr>
<tr><td>5</td><td colspan="8" rowspan="2">$4n-3$</td></tr>
<tr><td>≥ 6</td></tr>
</table>

Table 6: The planar Ramsey numbers $PR(K_m^-, K_n)$.

<table>
<tr><th>m \ n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>...</th></tr>
<tr><td>3</td><td>5</td><td>7</td><td>9</td><td>12</td><td>15</td><td colspan="3">$\geq 3n-6$</td></tr>
<tr><td>4</td><td colspan="8" rowspan="2">$4n-5$</td></tr>
<tr><td>≥ 5</td></tr>
</table>

Table 7: The planar Ramsey numbers $PR(K_m, K_n^-)$.

<table>
<tr><th>m \ n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>...</th></tr>
<tr><td>3</td><td colspan="8">$2n-3$</td></tr>
<tr><td>4</td><td>5</td><td>9</td><td>13</td><td colspan="5">?</td></tr>
<tr><td>5</td><td colspan="8" rowspan="2">$4n-5$</td></tr>
<tr><td>≥ 6</td></tr>
</table>

Table 8: The planar Ramsey numbers $PR(K_m^-, K_n^-)$.

<table>
<tr><th>m \ n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>...</th></tr>
<tr><td>3</td><td colspan="8">$3n-3$</td></tr>
<tr><td>4</td><td>7</td><td>10</td><td>13</td><td>17</td><td>20 | 21</td><td>23 | 24 | 25</td><td colspan="2">ineq. (3)</td></tr>
<tr><td>5</td><td colspan="8" rowspan="2">$4n-3$</td></tr>
<tr><td>⋮</td></tr>
</table>

Table 9: The planar Ramsey numbers $PR(C_m, K_n)$.

<table>
<tr><th>m \ n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>...</th></tr>
<tr><td>3</td><td>5</td><td>7</td><td>9</td><td>12</td><td>15</td><td colspan="3">$\geq 3n-6$</td></tr>
<tr><td>4</td><td>4</td><td>7</td><td>11</td><td>14</td><td colspan="4">ineq. (4)</td></tr>
<tr><td>5</td><td>5</td><td>9</td><td>13</td><td>17</td><td colspan="4">$\geq 4n-7$</td></tr>
<tr><td>6</td><td>6</td><td colspan="7" rowspan="3">$\geq 4n-6$</td></tr>
<tr><td>7</td><td rowspan="2">7</td></tr>
<tr><td>⋮</td></tr>
</table>

Table 10: The planar Ramsey numbers $PR(C_m, K_n^-)$.

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[1] K. Appel and W. Haken, *Every Planar Map is Four Colorable*, Illinois J. Math. 21 (1977), 429-490.

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[9] *http://www.mathcs.emory.edu/˜adudek/planram*