

On k -partite hypergraphs with the induced ε -density property

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Abstract

In this paper we extend the study of bipartite graphs with the induced ε -density property introduced by Frankl, Rödl, and the author. For a given k -partite k -uniform hypergraph \mathcal{G} we say that a k -partite k -uniform hypergraph $\mathcal{R} = (W_1, \dots, W_k, \mathcal{F})$ has the *induced ε -density property* if every subhypergraph of \mathcal{R} with at least $\varepsilon|\mathcal{F}|$ edges contains a copy of \mathcal{G} which is an induced subhypergraph of \mathcal{R} . We show that for every $\varepsilon > 0$ and positive integers k and n there exists a k -partite k -uniform hypergraph \mathcal{R} with the induced ε -density property for every $\mathcal{G} = (V_1, \dots, V_k, \mathcal{E})$ with $|V_1|, \dots, |V_k| \leq n$. We give several proofs of this result, some which allow for the hypergraph \mathcal{R} to be taken with at most $2^{2^{cn^{k-1}}}$ vertices.

1 Introduction

We say that R is an (*induced*) *Ramsey graph* of a given graph G if for any 2-coloring of the edges of R there is a monochromatic copy of G which is an induced subgraph of R . It is well-known that every graph G has a Ramsey graph R , as was proved by Erdős, Hajnal and Pósa [6], Deuber [3], and Rödl [14]. This variant of Ramsey's theorem immediately raises a numerical problem. For a given graph G , let $r_{ind}(G)$ denote the smallest integer m for which G has a Ramsey graph of order m . In 1975 Erdős and Rödl (see, *e.g.*, [2]) asked if $r_{ind}(G) < c^n$ for some absolute constant c and every graph G of order n . This problem remains open. Note that if true, the exponential upper bound in this question is best possible since it is known that $r_{ind}(K_n) \geq 2^{n/2}$. A weaker upper bound $r_{ind}(G) < 2^{cn(\log n)^2}$ was given by Kohayakawa, Prömel and Rödl [12]. A different proof of the same upper bound was recently found by Fox and

Sudakov [7]. Finally, let us mention that for G bipartite the answer to the Erdős and Rödl question is affirmative as was proved by Rödl [14].

Frankl, Rödl and the author considered in [4] a more general problem. For a given bipartite graph G we write $R \xrightarrow{\varepsilon} G$ and say that the bipartite graph $R = (W_1, W_2, F)$ has the *induced ε -density property* if every subgraph of R with at least $\varepsilon|F|$ edges contains a copy of G which is an induced subgraph of R . Clearly, if $R \xrightarrow{\varepsilon} G$ and $0 < \varepsilon \leq \frac{1}{2}$, then R is a Ramsey graph of G . Furthermore, the following holds.

Theorem 1.1 ([4]) *For every $\varepsilon > 0$ there is a constant $c = c(\varepsilon)$ such that for each positive integer n there exists a bipartite graph of order 2^{cn} such that $R \xrightarrow{\varepsilon} G$ for every bipartite graph $G = (V_1, V_2, E)$ with $|V_1|, |V_2| \leq n$.*

As one can see, by using standard probabilistic arguments, the exponential upper bound in Theorem 1.1 is best possible. Finally, note that setting $\varepsilon = \frac{1}{r}$ in Theorem 1.1 yields the following edge-coloring type result.

Corollary 1.2 *For every positive integer r there is a constant $c = c(r)$ such that for each positive integer n there exists a bipartite graph of order 2^{cn} such that any r -coloring of its edges yields a monochromatic and induced copy of every $G = (V_1, V_2, E)$ with $|V_1|, |V_2| \leq n$.*

In this paper we consider a similar question for hypergraphs. For a given k -partite k -uniform hypergraph \mathcal{G} we write $\mathcal{R} \xrightarrow{\varepsilon} \mathcal{G}$ and say that the k -partite k -uniform hypergraph $\mathcal{R} = (W_1, \dots, W_k, \mathcal{F})$ has the *induced ε -density property* if every subhypergraph of \mathcal{R} with at least $\varepsilon|\mathcal{F}|$ edges contains a copy of \mathcal{G} which is an induced subhypergraph of \mathcal{R} . First, we show that hypergraphs with the induced ε -density property exist.

Theorem 1.3 *For every $\varepsilon > 0$ and integers $k \geq 2$ and $n \geq 1$ there exists a k -partite k -uniform hypergraph \mathcal{R} such that $\mathcal{R} \xrightarrow{\varepsilon} \mathcal{G}$ for every $\mathcal{G} = (V_1, \dots, V_k, \mathcal{E})$ with $|V_1|, \dots, |V_k| \leq n$.*

The proof is based on an idea of Nešetřil and Rödl [13] combined with a density version of the Hales and Jewett theorem proved by Furstenberg and Katznelson [9].

Next, we will be interested in a quantitative extension of Theorem 1.3. In the special case when \mathcal{K} is a complete k -partite k -uniform hypergraph with the size of every partition class equal to n , Erdős [5] showed that there is a hypergraph \mathcal{R} of order $2^{cn^{k-1}}$, $c = c(\varepsilon, k)$, such that $\mathcal{R} \xrightarrow{\varepsilon} \mathcal{K}$. Here we show that in general the order of \mathcal{R} can be always bounded by $2^{2^{cn^{k-1}}}$.

Theorem 1.4 *For every $\varepsilon > 0$ and integer $k \geq 3$ there is a constant $c = c(\varepsilon, k)$ such that for each $n \geq 1$ there exists a k -partite k -uniform hypergraph \mathcal{R} of order $2^{2^{cn^{k-1}}}$ such that $\mathcal{R} \xrightarrow{\varepsilon} \mathcal{G}$ for every $\mathcal{G} = (V_1, \dots, V_k, \mathcal{E})$ with $|V_1|, \dots, |V_k| \leq n$.*

The proof is deterministic and gives an explicit construction of the hypergraph \mathcal{R} .

Note that by analogy with Corollary 1.2 setting $\varepsilon = \frac{1}{r}$ in Theorem 1.4 yields the following edge-coloring result.

Corollary 1.5 *For all integers $k \geq 3$ and $r \geq 1$ there is a constant $c = c(k, r)$ such that for each integer $n \geq 1$ there exists a k -partite k -uniform hypergraph of order $2^{2^{cn^{k-1}}}$ such that any r -coloring of its edges yields a monochromatic and induced copy of every $\mathcal{G} = (V_1, \dots, V_k, \mathcal{E})$ with $|V_1|, \dots, |V_k| \leq n$.*

2 The existence of hypergraphs with the induced ε -density property

Here we prove Theorem 1.3. The proof is essentially the same as the proof of Lemma 2 from [13] established by Nešetřil and Rödl (see also Chapter 5.3 in [10]). In [13] the authors used the Hales and Jewett theorem [11]. Here we will apply a density version of this result proved by Furstenberg and Katznelson [9].

We begin with notation. Let C_t^m be the m -cube over the alphabet $A = \{a_1, \dots, a_t\}$ of size $t \geq 1$ defined as

$$C_t^m = \{(x_1, \dots, x_m) : x_1, \dots, x_m \in A\}.$$

By a (combinatorial) line L in C_t^m we mean a set of t (suitably ordered) points such that $L = (\mathbf{p}_1, \dots, \mathbf{p}_t)$, $\mathbf{p}_i = (x_{i,1}, \dots, x_{i,m})$, $i = 1, \dots, t$, and there is a partition $[m] = M_1 \cup M_2$ with $M_1 \neq \emptyset$ satisfying

$$x_{i,j} = a_i \text{ for all } i = 1, \dots, t \text{ and all coordinates } j \in M_1,$$

and

$$x_{1,j} = x_{2,j} = \dots = x_{t,j} \text{ all coordinates } j \in M_2.$$

The well-known Hales and Jewett theorem [11] states that for every positive integers r and t there is a number $HJ(r, t)$ such that if $m \geq HJ(r, t)$ then any r -coloring of C_t^m yields a monochromatic line. Furstenberg and Katznelson [9] strengthened this result proving the following density version of the Hales and Jewett theorem.

Lemma 2.1 ([9]) *For every $\varepsilon > 0$ and a positive integer t there is a number $HJ(\varepsilon, t)$ such that if $m \geq HJ(\varepsilon, t)$ then any subset of C_t^m with cardinality at least $\varepsilon |C_t^m|$ contains a combinatorial line.*

Proof of Theorem 1.3. Fix $\varepsilon > 0$ and an integer $n \geq 1$. Let $\mathcal{H} = (U_1, \dots, U_k, \mathcal{I})$ be a k -partite k -uniform hypergraph (with suitable vertex sets U_1, \dots, U_k) which is a disjoint union of all k -partite k -uniform hypergraphs $\mathcal{G} = (V_1, \dots, V_k, \mathcal{E})$ with $|V_1|, \dots, |V_k| \leq n$ and no isolated vertices. We show that there is a k -partite k -uniform hypergraph $\mathcal{R} = (W_1, \dots, W_k, \mathcal{F})$ such that any of its subhypergraphs $\mathcal{R}' = (W_1, \dots, W_k, \mathcal{F}')$, $\mathcal{F}' \subseteq \mathcal{F}$, $|\mathcal{F}'| \geq \varepsilon |\mathcal{F}|$, contains a copy of \mathcal{H} which is an induced subhypergraph of \mathcal{R} . Since, \mathcal{H} contains an induced copy of every $\mathcal{G} = (V_1, \dots, V_k, \mathcal{E})$ with $|V_1|, \dots, |V_k| \leq n$, Theorem 1.3 will hold.

We define $\mathcal{R} = (W_1, \dots, W_k, \mathcal{F})$ as follows. Take as an alphabet (of size $t = |\mathcal{I}|$) the edge set of \mathcal{H} enumerated as $\mathcal{I} = \{I_1, \dots, I_t\}$. Let $m = HJ(\varepsilon, t)$ be the number guaranteed by Lemma 2.1. Let

$$W_l = \underbrace{U_l \times \dots \times U_l}_m = U_l^m$$

for each $l = 1, \dots, k$, and

$$\mathcal{F} = \left\{ F \in \binom{W_1 \cup \dots \cup W_k}{k} : \pi_j(F) \in \mathcal{I} \text{ for every } j = 1, \dots, m \right\},$$

where π_j denotes the j th projection, *i.e.*, if $F = \{(w_{1,1}, \dots, w_{1,m}), \dots, (w_{k,1}, \dots, w_{k,m})\}$ then $\pi_j(F) = \{w_{1,j}, \dots, w_{k,j}\}$. Note that $|\mathcal{F}| = |C_t^m| = t^m$ and define a bijection $\lambda : \mathcal{F} \rightarrow C_t^m$ such that

$$\lambda(F) = (\pi_1(F), \dots, \pi_m(F)),$$

for every $F \in \mathcal{F}$. This defines \mathcal{R} .

Let $\mathcal{R}' = (W_1, \dots, W_k, \mathcal{F}')$ be any subhypergraph of $\mathcal{R} = (W_1, \dots, W_k, \mathcal{F})$ such that $|\mathcal{F}'| \geq \varepsilon|\mathcal{F}|$. By Lemma 2.1, $\lambda(\mathcal{F}') \subseteq C_t^m$ contains a combinatorial line $L = (\mathbf{p}_1, \dots, \mathbf{p}_t)$, $\mathbf{p}_i = (x_{i,1}, \dots, x_{i,m}) \in \lambda(\mathcal{F}')$, $i = 1, \dots, t$, and there is a partition $[m] = M_1 \cup M_2$ with $M_1 \neq \emptyset$ satisfying

$$x_{i,j} = I_i \in \mathcal{I}$$

for every coordinate $j \in M_1$, and

$$x_{1,j} = x_{2,j} = \dots = x_{t,j} = I_{j'} \in \mathcal{I}$$

for every coordinate $j \in M_2$ and some j' depending only on j .

Now we show that the subhypergraph of \mathcal{R} induced on the set $\bigcup_{i=1}^t \lambda^{-1}(\mathbf{p}_i)$ is isomorphic to \mathcal{H} . To this end, we define for every $l = 1, \dots, k$ a mapping $\mu_l : U_l \rightarrow W_l$ as follows. For $u \in U_l$ let $\mu_l(u) = (w_1, \dots, w_m) \in W_l$, where

$$w_j = \begin{cases} u & \text{if } j \in M_1, \\ I_{j'} \cap U_l & \text{if } j \in M_2. \end{cases}$$

First, observe that since \mathcal{H} has no isolated vertices, $\bigcup_{l=1}^k \mu_l(U_l) = \bigcup_{i=1}^t \lambda^{-1}(\mathbf{p}_i)$. Now note that if $I = \{u_1, \dots, u_k\} \in \mathcal{I}$, then $F = \{\mu_1(u_1), \dots, \mu_k(u_k)\} \in \mathcal{F}$. Indeed, either $\pi_j(F) = I$ (if $j \in M_1$) or $\pi_j(F) = I_{j'}$ (if $j \in M_2$), and hence, $F \in \mathcal{F}$. On the other hand, if $I = \{u_1, \dots, u_k\} \notin \mathcal{I}$, then $F = \{\mu_1(u_1), \dots, \mu_k(u_k)\} \notin \mathcal{F}$ since $\pi_j(F) = I \notin \mathcal{I}$ for every $j \in M_1$. □

3 A quantitative extension of Theorem 1.3

Here we prove Theorem 1.4. The proof generalizes the approach taken in [4]. We will need the following auxiliary results. The first one was obtained independently by Sauer [15], Shelah [16], and Vapnik and Chervonenkis [17].

Lemma 3.1 ([15, 16, 17]) *Let \mathcal{P} be a family of subsets of an n -element set X satisfying*

$$|\mathcal{P}| > \sum_{i=0}^{k-1} \binom{n}{i}.$$

Then, there is a set $K \subseteq X$ of size k such that for every subset $S \subseteq K$ there is $P \in \mathcal{P}$ such that $S = P \cap K$.

The second auxiliary result, proved by Erdős [5], generalizes Zarankiewicz's theorem (see, e.g., [10]).

Lemma 3.2 ([5]) *For every integer $k \geq 2$ and $0 < \delta < 1$ there is a constant $c = c(k, \delta)$ such that every k -partite k -uniform hypergraph with the set of vertices $V_1 \cup \dots \cup V_k$, $|V_i| \leq n$, $i = 1, \dots, k$, and at least δn^k edges contains a copy of a complete k -partite k -uniform hypergraph $\mathcal{K}(X_1, \dots, X_k)$, where $|X_i| = \lfloor c^{k-1} \sqrt{\log_2 n} \rfloor$ for $i = 1, \dots, k$.*

Proof of Theorem 1.4. For given $\varepsilon > 0$ and $k \geq 3$ let $c_1 = c_1(\varepsilon, k)$ be a sufficiently large number so that

$$H(1/2) - H(1/2 - \varepsilon/4) > H(1/c_1^{k-1}), \quad (1)$$

where $H(p)$, $0 < p < 1$, is the binary entropy function defined as

$$H(p) = -p \log_2 p - (1-p) \log_2 (1-p).$$

Further, let $c_2 = c_2(k, \delta)$ be a constant which corresponds to $\delta = 1/c_1^{k-1}$ in Lemma 3.2. Finally, let $c = (k-1)(2/c_2)^{k-1}$. Clearly, $c = c(\varepsilon, k)$. Now for $n \geq 1$ define

$$N = 2^{\left(\frac{2}{c_2} n\right)^{k-1}} \quad \text{and} \quad M = \frac{N}{c_1},$$

and note that

$$N^{k-1} = 2^{(k-1)\left(\frac{2}{c_2} n\right)^{k-1}} = 2^{cn^{k-1}}.$$

We will always assume that n is sufficiently large and that $\varepsilon N^{k-1}/4$ is an integer.

Define $\mathcal{R} = (W_1, \dots, W_k, \mathcal{F})$ to be a k -partite k -uniform hypergraph with $|W_1| = \dots = |W_{k-1}| = N$ and $|W_k| = \binom{N^{k-1}}{N^{k-1}/2}$. Every vertex of W_k is added to all edges of precisely one $(k-1)$ -partite $(k-1)$ -uniform hypergraph on $W_1 \cup \dots \cup W_{k-1}$ with

$N^{k-1}/2$ edges as follows. Let τ be any bijection between W_k and the set of all $(k-1)$ -partite $(k-1)$ -uniform hypergraphs on $W_1 \cup \dots \cup W_{k-1}$ with $N^{k-1}/2$ edges. We define the edge set of \mathcal{R} as

$$\mathcal{F} = \bigcup_{w \in W_k} \{\{E \cup w\} : E \text{ is an edge in } \tau(w)\}.$$

Consequently, $\deg_{\mathcal{R}}(w) = N^{k-1}/2$ for every $w \in W_k$, and hence, $|\mathcal{F}| = \binom{N^{k-1}}{N^{k-1}/2} (N^{k-1}/2)$. Clearly,

$$|W_1| + \dots + |W_k| = (k-1)N + \binom{N^{k-1}}{N^{k-1}/2} \leq 2^{N^{k-1}} = 2^{2^{cn^{k-1}}}.$$

It remains to show that \mathcal{R} has the induced ε -density property for every hypergraph $\mathcal{G} = (V_1, \dots, V_k, \mathcal{E})$ with $|V_1|, \dots, |V_k| \leq n$.

Let $\mathcal{R}' = (W_1, \dots, W_k, \mathcal{F}')$ be a subhypergraph of \mathcal{R} such that

$$|\mathcal{F}'| \geq \varepsilon |\mathcal{F}| = \varepsilon \binom{N^{k-1}}{N^{k-1}/2} (N^{k-1}/2).$$

Then, there are at least $\frac{\varepsilon}{2} \binom{N^{k-1}}{N^{k-1}/2}$ vertices in W_k with degree $\deg_{\mathcal{R}'}(w) > \varepsilon N^{k-1}/4$. Otherwise, there would be at most $\frac{\varepsilon}{2} \binom{N^{k-1}}{N^{k-1}/2} - 1$ vertices in W_k of degree higher than $\varepsilon N^{k-1}/4$, which leads to the following contradiction

$$\begin{aligned} |\mathcal{F}'| &< \left(\frac{\varepsilon}{2} \binom{N^{k-1}}{N^{k-1}/2} - 1 \right) (N^{k-1}/2) + \binom{N^{k-1}}{N^{k-1}/2} (\varepsilon N^{k-1}/4) \\ &< \varepsilon \binom{N^{k-1}}{N^{k-1}/2} (N^{k-1}/2) = \varepsilon |\mathcal{F}| \leq |\mathcal{F}'|. \end{aligned}$$

Among the vertices (in W_k) of degree (in \mathcal{R}') higher than $\varepsilon N^{k-1}/4$ we find at least

$$\frac{\frac{\varepsilon}{2} \binom{N^{k-1}}{N^{k-1}/2}}{\frac{N^{k-1}}{2} - \frac{\varepsilon N^{k-1}}{4}} = \frac{\varepsilon}{\left(1 - \frac{\varepsilon}{2}\right) N^{k-1}} \binom{N^{k-1}}{N^{k-1}/2}$$

vertices of degree j , for some fixed j , $\varepsilon N^{k-1}/4 < j \leq N^{k-1}/2$. Let $U \subseteq W_k$ be the set of these vertices. Clearly, $\deg_{\mathcal{R}'}(u) = j$ for every $u \in U$, and

$$|U| \geq \frac{\varepsilon}{\left(1 - \frac{\varepsilon}{2}\right) N^{k-1}} \binom{N^{k-1}}{N^{k-1}/2}.$$

For $w \in W_k$ denote by $\mathcal{N}_{\mathcal{R}}(w)$ the $(k-1)$ -partite $(k-1)$ -uniform hypergraph on $W_1 \cup \dots \cup W_{k-1}$ generated by the edges of \mathcal{R} containing w , *i.e.*, E is an edge in $\mathcal{N}_{\mathcal{R}}(w)$ if and only if $\{E \cup w\}$ is an edge in \mathcal{R} . For $u \in U$ let $\mathcal{J}_u = \mathcal{N}_{\mathcal{R}}(u) \setminus \mathcal{N}_{\mathcal{R}'}(u)$.

Clearly, the hypergraph \mathcal{J}_u contains $N^{k-1}/2 - j$ edges for every $u \in U$. Consequently, there exists a hypergraph \mathcal{J} with $N^{k-1}/2 - j$ edges on $W_1 \cup \dots \cup W_{k-1}$ such that for at least

$$|U| / \binom{N^{k-1}}{\frac{N^{k-1}}{2} - j} > \frac{\varepsilon}{(1 - \frac{\varepsilon}{2}) N^{k-1}} \binom{N^{k-1}}{N^{k-1}/2} / \binom{N^{k-1}}{\frac{N^{k-1}}{2} - \frac{\varepsilon N^{k-1}}{4}} \quad (2)$$

vertices $u \in U$ the equality $\mathcal{J}_u = \mathcal{J}$ holds. Let $P = P(\mathcal{J})$ be the set of all these vertices.

Since for a fixed $p \in (0, 1)$ we have $\binom{N}{\lfloor pN \rfloor} = 2^{N(H(p)+o(1))}$ (see, *e.g.*, [1]), it follows from (2) and (1) that for $N = c_1 M$ large enough

$$\begin{aligned} |P| &> \frac{\varepsilon}{(1 - \frac{\varepsilon}{2}) N^{k-1}} \binom{N^{k-1}}{N^{k-1}/2} / \binom{N^{k-1}}{\frac{N^{k-1}}{2} - \frac{\varepsilon N^{k-1}}{4}} = 2^{N^{k-1}(H(1/2) - H(1/2 - \varepsilon/4) + o(1))} \\ &> 2^{N^{k-1}(H(1/c_1^{k-1}) + o(1))} = M^{k-1} \binom{N^{k-1}}{M^{k-1} - 1} \geq \sum_{i=0}^{M^{k-1}-1} \binom{N^{k-1}}{i}, \end{aligned} \quad (3)$$

and hence,

$$|P| > \sum_{i=0}^{M^{k-1}-1} \binom{N^{k-1} - j}{i}.$$

Let

$$\mathcal{P} = \{\mathcal{N}_{\mathcal{R}'}(u) : u \in P\} = \{\mathcal{N}_{\mathcal{R}}(u) \setminus \mathcal{J} : u \in P\}. \quad (4)$$

Note that $|\mathcal{P}| = |P|$. Lemma 3.1 yields that there exists a $(k-1)$ -partite $(k-1)$ -uniform hypergraph \mathcal{K} on $W_1 \cup \dots \cup W_{k-1}$ with M^{k-1} edges (disjoint with \mathcal{J}) such that for every subhypergraph $\mathcal{S} \subseteq \mathcal{K}$ there is a vertex $u \in P$ such that $\mathcal{S} = \mathcal{N}_{\mathcal{R}'}(u) \cap \mathcal{K}$.

Recall that $M^{k-1} = N^{k-1}/c_1^{k-1}$, $N = 2^{\binom{2}{c_2} n^{k-1}}$ and c_2 corresponds to $\delta = 1/c_1^{k-1}$ in Lemma 3.2. Hence, \mathcal{K} contains a complete $(k-1)$ -partite $(k-1)$ -uniform hypergraph

$$\mathcal{K}(X_1, \dots, X_{k-1}) \quad (5)$$

on a vertex set $X_1 \cup \dots \cup X_{k-1}$ such that

$$|X_1| = \dots = |X_{k-1}| = c_2^{-k-1} \sqrt[k-1]{\log_2 N} = 2n.$$

Moreover, note that

$$\mathcal{R}'[X_1 \cup \dots \cup X_{k-1} \cup P] = \mathcal{R}[X_1 \cup \dots \cup X_{k-1} \cup P]. \quad (6)$$

Now we are going to show that the subhypergraph of \mathcal{R}' induced on $W_1 \cup \dots \cup W_{k-1} \cup P$ contains an induced copy of every $\mathcal{G} = (V_1, \dots, V_k, \mathcal{E})$ with $|V_1|, \dots, |V_k| \leq n$. We may assume that $|V_1| = \dots = |V_k| = n$. For technical reasons, which will be

clarified later, it will be convenient to assume that no two vertices of V_k have the same neighborhood in \mathcal{G} . If this is not the case, then we can enlarge \mathcal{G} as follows. Let

$$U_i = V_i \cup \{u_{i,1}, \dots, u_{i,n}\} \text{ for each } i = 1, \dots, k-1,$$

$$U_k = V_k,$$

and

$$\mathcal{I} = \mathcal{E} \cup \{\{u_{1,j}, u_{2,j}, \dots, u_{k-1,j}, v_j\} : v_j \in V_k \text{ and } j = 1, \dots, n\}.$$

Let $\mathcal{H} = (U_1, \dots, U_k, \mathcal{I})$. Clearly, $|U_1| = \dots = |U_{k-1}| = 2n$, $|U_k| = n$, and \mathcal{H} contains an induced copy of \mathcal{G} . Therefore, in order to finish the proof of Theorem 1.4, it is enough to show that \mathcal{R}' contains an induced copy of \mathcal{H} .

We will find $U'_1 \subseteq W_1, \dots, U'_{k-1} \subseteq W_{k-1}$ and $U'_k \subseteq P$ such that the subhypergraph of \mathcal{R}' induced on $U'_1 \cup \dots \cup U'_k$ will be isomorphic to \mathcal{H} . Let $U'_1 \cup \dots \cup U'_{k-1}$ be the vertex set of the previous determined complete hypergraph $\mathcal{K}(X_1, \dots, X_{k-1})$ (cf. (5)), *i.e.*,

$$U'_i = X_i \text{ for every } i = 1, \dots, k-1.$$

It remains to determine U'_k . For every vertex $u \in U_k$ consider its neighborhood $\mathcal{N}_{\mathcal{H}}(u) \subseteq U_1 \cup \dots \cup U_{k-1}$ (and its isomorphic copy in $U'_1 \cup \dots \cup U'_{k-1}$). By the previous consideration there is a $w \in P$ such that $\mathcal{N}_{\mathcal{R}'}(w) \cap \mathcal{K}(X_1, \dots, X_{k-1})$ is isomorphic to $\mathcal{N}_{\mathcal{H}}(u)$. Let U'_k be the set of all those vertices w . Note that $|U'_k| = |U_k|$ since all vertices in U_k have different neighborhoods. This shows that $\mathcal{R}'[U'_1 \cup \dots \cup U'_k] = \mathcal{R}[U'_1 \cup \dots \cup U'_k]$ is isomorphic to \mathcal{H} (cf. (6)).

This completes the proof of Theorem 1.4. \square

4 An alternative perspective

The main result of this section can be used to handle the details in the proof of Theorem 1.4 slightly differently. Moreover, this result might be also of some interest.

For disjoint sets X_1, \dots, X_k with $|X_i| = x_i$, $i = 1, \dots, k$, denote by $\mathcal{K}(X_1, \dots, X_k)$ the complete k -partite k -uniform hypergraph on a vertex set $X_1 \cup \dots \cup X_k$. Further let $m(x_1, \dots, x_k, \mathcal{K}_0)$ denote the total number of subhypergraphs of $\mathcal{K}(X_1, \dots, X_k)$ without a copy of the fixed hypergraph \mathcal{K}_0 .

Theorem 4.1 *Let \mathcal{K}_0 be a fixed complete k -partite k -uniform hypergraph with size of partition classes n_1, \dots, n_k . Moreover, let $\mathcal{F}_1, \dots, \mathcal{F}_m \subseteq \mathcal{K}(X_1, \dots, X_k)$ be distinct k -partite k -uniform hypergraphs with $m > m(x_1, \dots, x_k, \mathcal{K}_0)$ and $|X_i| = x_i$ for $i = 1, \dots, k$. Then, there exists a copy of $\mathcal{K}_0 \subseteq \mathcal{K}(X_1, \dots, X_k)$ such that all of its subhypergraphs \mathcal{S} occur as induced in some \mathcal{F}_j , $1 \leq j \leq m$, *i.e.*, $\mathcal{S} = \mathcal{K}_0 \cap \mathcal{F}_j$.*

Proof. For the proof we consider the \mathcal{F}_j as subsets of the edge set of $\mathcal{K}(X_1, \dots, X_k)$. Next we recall some definitions for families of sets.

Let X be an n -set and $\mathcal{F} \subseteq 2^X$. For $Y \subseteq X$, define the *trace* of \mathcal{F} on Y as

$$\mathcal{F}|_Y = \{F \cap Y : F \in \mathcal{F}\}.$$

For $x \in X$ define the *down-shift* D_x by

$$D_x(\mathcal{F}) = \{D_x(F) : F \in \mathcal{F}\},$$

where

$$D_x(F) = \begin{cases} F \setminus \{x\} & \text{if } x \in F \text{ and } (F \setminus \{x\}) \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

Clearly, $|\mathcal{F}| = |D_x(\mathcal{F})|$ for all $x \in X$. Furthermore, it is known due to Frankl [8] that the down-shifting does not increase the trace, *i.e.*, for every $Y \subseteq X$ and $x \in X$,

$$|D_x(\mathcal{F})|_Y \leq |\mathcal{F}|_Y \quad (7)$$

(for details, see the proof of Theorem 1 in [8]).

Now we can proceed with the proof of Theorem 4.1. Suppose that $\mathcal{F}_1, \dots, \mathcal{F}_m$ never produce all induced subhypergraphs of \mathcal{K}_0 . That means for $\mathcal{G} = \{\mathcal{F}_1, \dots, \mathcal{F}_m\}$ and all choices of $Y_i \subseteq X_i$, $|Y_i| = n_i$, $i = 1, \dots, k$,

$$|\mathcal{G}|_{\mathcal{K}(Y_1, \dots, Y_k)} < 2^{n_1 \cdots n_k}, \quad (8)$$

since $2^{n_1 \cdots n_k}$ is the total number of all subhypergraphs of \mathcal{K}_0 . By (7) for each edge E in $\mathcal{K}(X_1, \dots, X_k)$ the down-shift $D_E(\mathcal{G})$ will produce a new family $\{D_E(\mathcal{F}_j) : j = 1, \dots, m\}$ of k -partite k -uniform hypergraphs satisfying (8). By iterating D_E for all E in $\mathcal{K}(X_1, \dots, X_k)$, eventually we end up with a family $\tilde{\mathcal{G}}$ of sets $\{\tilde{\mathcal{F}}_1, \dots, \tilde{\mathcal{F}}_m\}$, where $\tilde{\mathcal{G}}$ is hereditary, *i.e.*, if $\mathcal{E} \subseteq \mathcal{F} \in \tilde{\mathcal{G}}$ then $\mathcal{E} \in \tilde{\mathcal{G}}$. Now all we have to note is that no $\tilde{\mathcal{F}} \in \tilde{\mathcal{G}}$ can contain a $\mathcal{K}(Y_1, \dots, Y_k)$, because of heredity this $\mathcal{K}(Y_1, \dots, Y_k)$ and all its subhypergraphs would be members of $\tilde{\mathcal{G}}$. This shows that $m \leq m(x_1, \dots, x_k, \mathcal{K}_0)$. But by assumption $m > m(x_1, \dots, x_k, \mathcal{K}_0)$, a contradiction. \square

Remark 4.2 It is easy to see that Theorem 4.1 generalizes Lemma 3.1. Indeed, for $n \geq k$ let X be the edge set of a complete bipartite graph $\mathcal{K}(X_1, X_2) = K_{1,n}$ with $|X_1| = 1$ and $|X_2| = n$ (clearly, $|X| = n$). Define $\mathcal{K}_0 = K_{1,k}$ as a complete bipartite graph with size of partition classes 1 and k . Note that

$$m(1, n, K_{1,k}) = \sum_{i=0}^{k-1} \binom{n}{i}. \quad (9)$$

Let \mathcal{P} be any family of subsets of X (*i.e.*, subgraphs of $K_{1,n}$). By assumption (of Lemma 3.1),

$$|\mathcal{P}| > \sum_{i=0}^{k-1} \binom{n}{i},$$

and hence by (9),

$$|\mathcal{P}| > m(1, n, K_{1,k}).$$

Thus, Theorem 4.1 implies that there is a copy of $K_{1,k}$ in X , say K , such that all of its subgraphs (subsets of K) occur as induced. Consequently, Lemma 3.1 holds.

Now we show how one can prove Theorem 1.4. Let \mathcal{K}_0 be a fixed complete $(k-1)$ -partite $(k-1)$ -uniform hypergraph with size of every partition class $2n$. Following the notation from the proof of Theorem 1.4 note that every element of \mathcal{P} (cf. (4)) is in $\mathcal{K}(W_1, \dots, W_{k-1})$. Let us assume that

$$|\mathcal{P}| > m(|W_1|, \dots, |W_{k-1}|, \mathcal{K}_0). \quad (10)$$

Then, by Theorem 4.1, there is a copy of \mathcal{K}_0 in $\mathcal{K}(W_1, \dots, W_{k-1})$ such that all of its subhypergraphs occur as induced. The rest of the proof is the same.

It remains to show that (10) holds. Determining the precise value of $m(|W_1|, \dots, |W_{k-1}|, \mathcal{K}_0)$ is not an easy problem. Here we find an obvious upper bound based on Lemma 3.2. Recall that $\delta = 1/c_1^{k-1}$. By Lemma 3.2 we infer that the number of hypergraphs in $\mathcal{K}(W_1, \dots, W_{k-1})$ with no \mathcal{K}_0 is smaller than

$$\sum_{j=0}^{\delta N^{k-1}} \binom{N^{k-1}}{j} = 2^{N^{k-1}(H(\delta)+o(1))} = 2^{N^{k-1}(H(1/c_1^{k-1})+o(1))} < |\mathcal{P}|,$$

where the last inequality follows from (3). Hence, (10) holds, as required.

5 Concluding remarks

In this paper we have established an upper bound on the size of k -partite k -uniform hypergraphs with the induced ε -density property. We conjecture that Theorem 1.4 can be further strengthened by replacing $2^{2^{cn^{k-1}}}$ by $2^{cn^{k-1}}$.

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