A note on the H-free process for disconnected H

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Abstract

Let H be a fixed graph. The H-free process is the random graph process which starts with an empty graph and adds edges one at a time chosen uniformly at random from the collection of pairs of vertices that neither appear as edges in the previous steps nor form a copy of H. The process terminates with a maximal H-free graph. In this note we are interested in determining the number of edges of such a maximal graph in the case when H is the vertex disjoint union of different graphs.

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1 Introduction

For a given graph H consider the following H-free random graph process. We begin with the empty graph on n vertices denoted by G_0. At step i we form the graph G_i by adding an edge to G_{i-1} chosen uniformly at random from the collection of pairs of vertices that neither appear as edges in G_{i-1} nor form a copy of H. The process terminates with a maximal H-free graph on n vertices denoted by G_t. Determining the structural properties of G_t (such as a number of edges in G_t for example) is a basic question in this field (see, e.g., [1, 2, 5, 6, 7]).

Let the random variable $M_n(H)$ be the number of edges in G_t. Osthus and Taraz [6] showed that if $H = (V, E)$ is strictly 2-balanced then asymptotically almost surely (a.a.s. in short)

$$\Omega\left(n^{2-\frac{|V(H)|-2}{|E(H)|-1}}\right) \le M_n(H) \le O\left((\log n)^{1/(|\Delta(H)|-1)} \cdot n^{2-\frac{|V(H)|-2}{|E(H)|-1}}\right). \tag{1}$$

Recently, Bohman [1] precisely determined $M_n(K_3) = \Theta(\sqrt{\log n} \cdot n^{3/2})$ proving a conjecture of Spencer [8]. Subsequently, Bohman and Keevash [2] improved the lower bound from (1) and showed that if $H = (V, E)$ is strictly 2-balanced then a.a.s.

$$\Omega\left((\log n)^{1/(|E(H)|-1)} \cdot n^{2-\frac{|V(H)|-2}{|E(H)|-1}}\right) \leq M_n(H).$$

They also conjectured that the upper bound has the same order of magnitude as the above lower bound. As a matter of fact, the conjecture holds for cycles [3].

In this note we are interested in determining $M_n(H)$ in the case when H is the disjoint union of graphs. Denote by $m(G)$ the ratio of the number of edges to the number of vertices in the densest subgraph of G, that is,

$$m(G) = \max\left\{\frac{|E(F)|}{|V(F)|} : \emptyset \neq F \subseteq G\right\}.$$

The graph G is called *strictly balanced* if $m(F) < m(G)$ whenever $F \subsetneq G$. Moreover, let $\mathrm{aut}(G)$ be the number of automorphisms of G.

Theorem 1.1 *Let H be the vertex disjoint union of two different graphs H_1 and H_2.*

(i) If $m(H_1) > m(H_2)$ then $\lim_{n\to\infty} \Pr[M_n(H) = M_n(H_1)] = 1$.

(ii) If H_1 and H_2 are strictly balanced and $m(H_1) = m(H_2)$ then there exists a constant γ, $0 \leq \gamma \leq 1$, such that

$$\lim_{n\to\infty} \Pr[M_n(H) = M_n(H_1)] = \gamma$$

and

$$\lim_{n\to\infty} \Pr[M_n(H) = M_n(H_2)] = 1 - \gamma.$$

Moreover, constant γ can be computed explicitly as

$$\gamma = \int\int_{x\geq y\geq 0} \frac{|E(H_1)|}{\mathrm{aut}(H_1)} x^{|E(H_1)|-1} \exp\left\{-\frac{x^{|E(H_1)|}}{\mathrm{aut}(H_1)}\right\} \cdot \frac{|E(H_2)|}{\mathrm{aut}(H_2)} (x-y)^{|E(H_2)|-1} \exp\left\{-\frac{(x-y)^{|E(H_2)|}}{\mathrm{aut}(H_2)}\right\} \, dx\, dy.$$

(Part (i) was already observed by Osthus and Taraz [6].) In particular, for cycles we obtain the following corollary.

Corollary 1.2 *Let H be the vertex disjoint union of two cycles C_k and C_ℓ, $3 \leq k < \ell$, and*

$$\gamma = \int\int_{x\geq y\geq 0} \frac{x^{k-1}}{2} \exp\left\{-x^k/(2k)\right\} \frac{(x-y)^{\ell-1}}{2} \exp\left\{-(x-y)^\ell/(2\ell)\right\} \, dx\, dy.$$

Then,

$$\lim_{n\to\infty} \Pr[M_n(H) = M_n(C_k)] = \gamma$$

and

$$\lim_{n\to\infty} \Pr[M_n(H) = M_n(C_\ell)] = 1 - \gamma.$$

For example, if H is the disjoint union of C_3 and C_4 then the corresponding constant (computed with Mathematica) is $\gamma = 0.545744...$, and hence by a new result of Bohman and Keevash [3],

$$\lim_{n\to\infty} \Pr[M_n(H) = \Theta(\sqrt{\log n} \cdot n^{3/2})] = 0.545744...$$

and

$$\lim_{n\to\infty} \Pr[M_n(H) = \Theta(\sqrt[3]{\log n} \cdot n^{4/3})] = 0.454256...$$

One can easily extend Theorem 1.1 as follows.

Theorem 1.3 *Let H be the vertex disjoint union of k different graphs $H_1, \dots, H_k$ for $k \ge 2$.*

- *(i) If there is an i, $1 \le i \le k$, such that $m(H_i) > m(H_j)$ for all $j \ne i$ then* $\lim_{n\to\infty} \Pr[M_n(H) = M_n(H_i)] = 1$.
- *(ii) If $H_1, \dots, H_k$ are strictly balanced and $m(H_1) = \cdots = m(H_k)$ then there are numbers $\gamma_1, \dots, \gamma_k$, $0 \le \gamma_i \le 1$, such that $\gamma_1 + \cdots + \gamma_k = 1$ and* $\lim_{n\to\infty} \Pr[M_n(H) = M_n(H_i)] = \gamma_i$ *for every $i = 1, \dots, k$.*

It is also possible to determine numbers γ_i. We omit the details here since this is more technical and it would serve only to mask the main ideas.

2 Proof of Theorem 1.1

We start with the proof of (i). Let H be the disjoint union of two different graphs H_1 and H_2 satisfying $m(H_1) > m(H_2)$. Given an integer M, $0 \le M \le \binom{n}{2}$, denote by $\mathbb{G}(n, M)$ the *uniform random graph* of order n with exactly M edges. It is well-known that the threshold for containment of G in $\mathbb{G}(n, M)$ is $n^{2-1/m(G)}$ (see, e.g., Corollary 4.14 in [4]). Consequently, a.a.s. a copy of H_2 appears before H_1. After that step the process behaves like the H_1-free process. This completes the proof of (i).

Now we are going to show (ii). Let H be the disjoint union of two different strictly balanced graphs H_1 and H_2 satisfying $m(H_1) = m(H_2)$. The H-free process on n vertices can be also viewed as follows. We start with the empty graph G_0 of order n and a random permutation σ of $\{1, \dots, \binom{n}{2}\}$. Denote by $\{e_1, \dots, e_{\binom{n}{2}}\}$ the set of all pairs of vertices of G_0. At step i, $1 \le i \le \binom{n}{2}$, we form the graph G_i by adding the edge $e_{\sigma(i)}$ to G_{i-1} if G_i does not contain a copy of H; otherwise $G_i = G_{i-1}$. Let X_1 be a random variable such that X_1 equals i if at step i:

(i) the graph G_{i-1} is H_1-free with $E(G_{i-1}) = \{e_{\sigma(1)}, \ldots, e_{\sigma(i-1)}\}$, and

(ii) adding $e_{\sigma(i)}$ to G_{i-1} forms a copy of H_1.

In other words, X_1 counts the number of edges in the process until the first possible copy of H_1 can be created. Similarly, we define X_2 with respect to H_2.

First we show that

$$\lim_{n\to\infty} \Pr[X_1 = X_2] = 0. \tag{2}$$

Note that if $X_1 = X_2$ then both a copy of H_1 and H_2 appear simultaneously. Let F be the union of those copies (clearly not edge disjoint union) and F' their intersection. Thus,

$$|V(F)| = |V(H_1)| + |V(H_2)| - |V(F')| \text{ and } |E(F)| = |E(H_1)| + |E(H_2)| - |E(F')|.$$

Let $m = m(H_1) = m(H_2)$. Hence,

$$|E(H_1)| = m|V(H_1)| \text{ and } |E(H_2)| = m|V(H_2)|.$$

Moreover, since H_1 and H_2 are strictly balanced, F' is not isomorphic neither to H_1 nor to H_2. This implies that F' is a proper subgraph of a copy of H_1 (and also H_2) in F. Thus,

$$|E(F')| < m|V(F')|.$$

Consequently, the density of F satisfies

$$\begin{aligned} m(F) \geq \frac{|E(F)|}{|V(F)|} &= \frac{|E(H_1)| + |E(H_2)| - |E(F')|}{|V(H_1)| + |V(H_2)| - |V(F')|} \\ &> \frac{m(|V(H_1)| + |V(H_2)| - |V(F')|)}{|V(H_1)| + |V(H_2)| - |V(F')|} = m = m(H_1) = m(H_2) \end{aligned}$$

and so in our process a.a.s. a copy of H_1 (or a copy H_2) appears before F, that is, (2) holds.

Next we observe that once a copy of H_2 appears before a copy of H_1 the process behaves like the H_1-free process. Conversely, if a copy of H_1 appears before a copy of H_2 then the process becomes the H_2-free process. Hence, by (2) we obtain

$$\lim_{n\to\infty} \Pr[M_n(H) = M_n(H_1)] = \lim_{n\to\infty} \Pr[X_1 > X_2] \tag{3}$$

and

$$\lim_{n\to\infty} \Pr[M_n(H) = M_n(H_2)] = \lim_{n\to\infty} \Pr[X_2 > X_1]. \tag{4}$$

Setting $\gamma = \lim_{n\to\infty} \Pr[X_1 > X_2]$ completes the proof of (ii).

In order to determine constant γ we need the following well-known result. Let $G = (V, E)$ be a strictly balanced graph and $M = (c/2)n^{2-1/m(G)}$, $c > 0$. Then,

$$\lim_{n\to\infty} \Pr[\mathbb{G}(n, M) \not\supset G] = \exp\left\{-\frac{c^{|E(G)|}}{\mathrm{aut}(G)}\right\} \tag{5}$$

(see, e.g., Corollary 4.2 in [4]). Hence, for $M = (c/2)n^{2-1/m(H_1)} = (c/2)n^{2-1/m(H_2)}$, $c > 0$, we get

$$\Pr[X_1 \le M] = \Pr[\mathbb{G}(n,M) \supset H_1] = 1 - \Pr[\mathbb{G}(n,M) \not\supset H_1],$$

and so by (5),

$$\lim_{n\to\infty} \Pr[X_1 \le M] = 1 - \exp\left\{-\frac{c^{|E(H_1)|}}{\mathrm{aut}(H_1)}\right\}. \tag{6}$$

Similarly,

$$\lim_{n\to\infty} \Pr[X_2 \le M] = 1 - \exp\left\{-\frac{c^{|E(H_2)|}}{\mathrm{aut}(H_2)}\right\}. \tag{7}$$

Now we define two auxiliary random variables

$$Y_1 = \frac{cX_1}{M} = \frac{2X_1}{n^{2-1/m(H_1)}} \quad \text{and} \quad Y_2 = \frac{cX_2}{M} = \frac{2X_2}{n^{2-1/m(H_2)}}.$$

Let $F_1(c) = \Pr[Y_1 \le c]$ and $F_2(c) = \Pr[Y_2 \le c]$ be their probability density functions. Note that (6) and (7) yield

$$\lim_{n\to\infty} F_1(c) = 1 - \exp\left\{-\frac{c^{|E(H_1)|}}{\mathrm{aut}(H_1)}\right\} \quad \text{and} \quad \lim_{n\to\infty} F_2(c) = 1 - \exp\left\{-\frac{c^{|E(H_2)|}}{\mathrm{aut}(H_2)}\right\}.$$

Moreover, the corresponding probability mass functions $f_1(c) = F_1'(c)$ and $f_2(c) = F_2'(c)$ satisfy,

$$\lim_{n\to\infty} f_1(c) = \frac{|E(H_1)|}{\mathrm{aut}(H_1)} c^{|E(H_1)|-1} \exp\left\{-\frac{c^{|E(H_1)|}}{\mathrm{aut}(H_1)}\right\} \tag{8}$$

and

$$\lim_{n\to\infty} f_2(c) = \frac{|E(H_2)|}{\mathrm{aut}(H_2)} c^{|E(H_2)|-1} \exp\left\{-\frac{c^{|E(H_2)|}}{\mathrm{aut}(H_2)}\right\}. \tag{9}$$

Thus,

$$\begin{aligned}\lim_{n\to\infty} \Pr[X_1 > X_2] = \lim_{n\to\infty} \Pr[cX_1/M > cX_2/M] &= \lim_{n\to\infty} \Pr[Y_1 > Y_2] \\ &= \lim_{n\to\infty} \int\int_{x>y\ge 0} f_1(x) f_2(x-y)\, dx\, dy,\end{aligned}$$

and consequently, by (2), (3), (4), (8), and (9) the theorem holds.

3 Concluding remarks

In this note we determined the number of edges in the $(H_1 \cup H_2)$-free process, where H_1 and H_2 are vertex disjoint graphs. In fact, we showed that for H_1 and H_2 strictly balanced satisfying $m(H_1) = m(H_2)$ the process terminates with a maximal $(H_1 \cup H_2)$-free graph being either a maximal H_1-free graph (with probability close to γ) or a maximal H_2-free graph (with probability close to $1-\gamma$). Therefore, any structural property of a maximal graph arising from the $(H_1 \cup H_2)$-free process can be derived from the H_1 and H_2-free processes.

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