New Upper Bound on Vertex Folkman Numbers

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Abstract. In 1970, J. Folkman proved that for a given integer r and a graph G of order n there exists a graph H with the same clique number as G such that every r coloring of vertices of H yields at least one monochromatic copy of G. His proof gives no good bound on the order of graph H, i.e. the order of H is bounded by an iterated power function. A related problem was studied by Łuczak, Ruciński and Urbański, who gave some explicite bound on the order of H when G is a clique. In this note we give an alternative proof of Folkman's theorem with the relatively small order of H bounded from above by $O(n^3 \log^3 n)$. This improves Łuczak, Ruciński and Urbański's result.

Key words: Ramsey theory, vertex Folkman numbers

1 Introduction

For a given graph $H = (V, E)$, let $c : V(H) \to \{1, \ldots, r\}$ be an r-coloring of vertices of H. We write $H \to (G)^v_r$ (or $H \underset{ind}{\longrightarrow} (G)^v_r$) if for every r-coloring c of vertices of H, there exists a copy of G (or an induced copy), say G', such that $V(G') \subseteq c^{-1}(i)$, for some color $i \in \{1, \ldots, k\}$. Moreover, let $cl(G)$ be the *clique number* of G, i.e. the order of a maximal clique in G. In [4], J. Folkman proved that for every graph G there exists a graph H such that $H \to (G)^v_r$ and $cl(H) = cl(G)$. Clearly $cl(H) \geq cl(G)$ for any graph with $H \to (G)^v_r$ and thus Folkman's theorem is in this sense the best possible. For $G = K_k$, i.e. for a clique of size k, a related question was studied e.g. in [6, 8]. For positive integers r, k and l with $k < l$ the *vertex Folkman number* is

$$F(r, k, l) = \min \left\{ |V(H)| \;\middle|\; H \to (K_k)^v_r \text{ and } cl(H) = l - 1 \right\}.$$

Clearly Folkman's theorem yields that $F(r, k, l)$ is well-defined for any $k < l$. Determining the precise value of $F(r, k, l)$ is not an easy problem in general. Only few of these numbers are known and mostly they were found with the aid of computers (see e.g. [2]). Some special cases were considered in [6, 8]. Obviously, the most restrictive and challenging case is to determine the exact value of $F(r, k, k+1)$ (or more realistic to estimate it). The upper bound on this number, based on Folkman's proof [4], is an iterated power function. Łuczak, Ruciński and Urbański [8] improved this bound and showed that for instance for 2 colors $F(2,k,k+1) = O(k!)$. In this note we will prove a relatively small upper bound on $F(r,k,l)$ (see comments after Theorem 1). In fact, we will prove a more general statement. In 1991, Brown and the second author [1] showed that for every natural number r there are constants c_1 and c_2 such that

$$c_1 n^2 \leq \max\left\{ \min\left\{ |V(H)| \;\middle|\; H \xrightarrow[ind]{} (G)^v_r \right\} \right\} \leq c_2 n^2 \log^2 n, \tag{1}$$

where the maximum is taken over all graphs G of order n. In this note we will not only enforce $H \rightarrow (G)^v_r$, but also require $cl(H) = cl(G)$. Following the idea from [1], we will show that adding this new constraint will increase the upper bound in (1) only by a factor of $n \log n$.

Theorem 1 *For a given natural number r there exists a constant $c = c(r)$ such that for every graph G of order n the following inequality holds*

$$\min\left\{ |V(H)| \;\middle|\; H \xrightarrow[ind]{} (G)^v_r \text{ and } cl(H) = cl(G) \right\} \leq cn^3 \log^3 n.$$

For $G = K_k$ Theorem 1 immediately yields $F(r,k,l) \leq ck^3 \log^3 k$. In fact, modifying the proof of Theorem 1, one can show a stronger result, which we state below without proof.

Theorem 2 *For a given natural number r and an arbitrarily small $\varepsilon > 0$ there exists a constant $c = c(r, \varepsilon)$ such that for any natural numbers k and l with $k < l$ the vertex Folkman number satisfies*

$$F(r,k,l) \leq ck^{2+\varepsilon}.$$

We were not able to find any nontrivial lower bound on $F(r,k,l)$. It would be interesting to decide if for every r the ratio $\frac{F(r,k,k+1)}{k}$ tends to infinity as k tends to infinity. This work is currently in progress.

2 Generalized Quadrangles

In this section we describe some basic properties of generalized quadrangles, which we use to prove Theorem 1.

A *generalized quadrangle* (see e.g. [5]) is an incidence structure of a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of lines such that:

(i) any two points are on at most one line,
(ii) if p is a point not on a line ℓ, then there is a unique point $\mathsf{p}' \in \ell$ collinear with p, and hence, no three lines form a triangle,
(iii) every line contains $q+1$ points, and every point lies on $q+1$ lines.

It is known that for every prime power q such incidence structure $\mathcal{Q}$ exists with $|\mathcal{P}| = |\mathcal{L}| = q^3 + q^2 + q + 1$ (see e.g. [7,9]). Let $\mathcal{Q}_I = (\mathcal{P}, \mathcal{L}, \mathcal{E})$ be a bipartite graph, which corresponds to the above incidence structure $\mathcal{Q}$, with the set of vertices $\mathcal{P} \cup \mathcal{L}$ and the set of edges $\mathcal{E} = \{(\mathsf{p}, \ell) \in \mathcal{P} \times \mathcal{L} \mid \mathsf{p} \text{ lies on line } \ell\}$. Note that $\mathcal{Q}_I$ is a $(q+1)$-regular graph. For a given graph G and disjoint subsets B and C of vertices of G we denote by $e(B,C)$ the number of edges that connect a vertex of B with a vertex of C. The following statement is an easy consequence of the fact that the graph $\mathcal{Q}_I$ is $(q+1)$-regular.

Fact 3 *Let $\mathcal{Q}_I = (\mathcal{P}, \mathcal{L}, \mathcal{E})$ be the bipartite graph defined above with $|\mathcal{P}| = |\mathcal{L}| = N$. Suppose that $Y \subseteq \mathcal{P}$ with $|Y| = \alpha N$, for some $0 < \alpha < 1$. Then,*

$$e(Y, \mathcal{L}) = \alpha N^{\frac{4}{3}}(1 + o(1)).$$

3 Proof of Theorem 1

As we mentioned in the introduction, the proof of Theorem 1 goes along the lines of the proof of Theorem 2.2 from [1].

Fix a natural number r, $r \geq 2$ (for $r = 1$ Theorem 1 holds trivially). Let α be a real number satisfying $0 < \alpha \leq \frac{1}{2}$. Let $G = (V, E)$ be a graph of order n with $V = \{v_1, \ldots, v_n\}$. We always assume that n is sufficiently large. We will show that there exists a graph H of order $cn^3 \log^3 n$, $c = c(r)$, such that $cl(H) = cl(G)$ and any subgraph of H induced by a set of cardinality $\lfloor \alpha |V(H)| \rfloor$ contains an induced copy of G. For $\alpha = \frac{1}{r}$ this will obviously imply the statement of Theorem 1.

By Bertrand's postulate we know that for any number $z \geq 1$ there exists a prime number between z and $2z$. In particular for a given n there is a prime q such that

$$\frac{4}{\alpha} n \log n \leq q + 1 \leq \frac{8}{\alpha} n \log n.$$

For such q let $t = q + 1$ and let x be such that

$$t = xn + m, \tag{2}$$

where $0 \leq m < n$. Consequently,

$$\frac{3}{\alpha} \log n \leq x \leq \frac{8}{\alpha} \log n. \tag{3}$$

Let $\mathcal{Q}$ be a generalized quadrangle from the previous section with $|\mathcal{P}| = |\mathcal{L}| = N$, where $N = q^3 + q^2 + q + 1$. We construct a "random graph" H with the vertex set $\mathcal{P}$ as follows. In view of (2) one can partition each line into sets of size x and $x + 1$, respectively. For each line ℓ we choose one such ordered partition $\ell_1, \ldots, \ell_n$ randomly and uniformly from the sets of all

$$\gamma = \frac{t!}{\left(x!\right)^{n-m} \left((x+1)!\right)^m} \tag{4}$$

partitions. For each $\mathsf{u} \in \ell_i$ and $\mathsf{w} \in \ell_j$ we join $\{\mathsf{u}, \mathsf{w}\}$ by an edge if and only if $\{v_i, v_j\} \in E(G)$. Note that H is well-defined because of condition (i). Moreover, condition (ii) yields that $cl(H) = cl(G)$. In fact, more is true: every triangle of H is contained entirely within some ℓ. Observe that there are γ^N graphs H constructed this way. We will show that the graph randomly chosen from the space of all such graphs has the following property. Every set $Y \subseteq V(H)$, $|Y| = \lfloor \frac{1}{r} N \rfloor$ induces a subgraph $H[Y]$, which contains G as an induced subgraph.

For $Y \subseteq V(H)$ with cardinality $|Y| = \lfloor \frac{1}{r} N \rfloor = \lfloor \alpha N \rfloor$ let A_Y be the event that G is an induced subgraph of $H[Y]$. For each $\ell \in \mathcal{L}$ let A_ℓ be the event that some ℓ_i in the partition of ℓ is disjoint from Y. Note that if A_Y fails, then all events A_ℓ, $\ell \in \mathcal{L}$, must occur. Consequently,

$$\bar{A}_Y \subseteq \bigcap_{\ell \in \mathcal{L}} A_\ell.$$

Furthermore, since all events A_ℓ are independent we obtain

$$\Pr(\bar{A}_Y) \le \prod_{\ell \in \mathcal{L}} \Pr(A_\ell). \tag{5}$$

For each $\ell \in \mathcal{L}$ we will bound from above the probability $\Pr(A_\ell)$. Recall that $|\ell| = t$ and γ is the number of all ordered partitions of ℓ into n sets ($n - m$ of size x and m of size $x + 1$). Let $|Y \cap \ell| = y_\ell$. For a fixed i, the number of partitions of ℓ for which ℓ_i, $|\ell_i| = x$, is disjoint from Y is at most

$$\binom{t - y_\ell}{x} \frac{(t-x)!}{\big(x!\big)^{n-m-1}\big((x+1)!\big)^m} = \frac{\binom{t-y_\ell}{x}}{\binom{t}{x}}\gamma. \tag{6}$$

Similarly, the number of partitions of ℓ for which ℓ_i, $|\ell_i| = x + 1$, is disjoint from Y is at most

$$\binom{t - y_\ell}{x+1} \frac{(t-x-1)!}{\big(x!\big)^{n-m}\big((x+1)!\big)^{m-1}} = \frac{\binom{t-y_\ell}{x+1}}{\binom{t}{x+1}}\gamma. \tag{7}$$

Hence, (4), (6) and (7) yield

$$\Pr(A_\ell) \le (n-m)\frac{\binom{t-y_\ell}{x}}{\binom{t}{x}} + m\frac{\binom{t-y_\ell}{x+1}}{\binom{t}{x+1}} \le (n-m)\mathrm{e}^{\frac{-xy_\ell}{t}} + m\mathrm{e}^{\frac{-(x+1)y_\ell}{t}} \le n\mathrm{e}^{\frac{-xy_\ell}{t}}.$$

The last inequality holds, since for for any natural numbers a, b, c with $a - b \ge c$, the following is true

$$\frac{\binom{a-b}{c}}{\binom{a}{c}} = \frac{(a-b-c+1)\cdots(a-b)}{(a-c+1)\cdots a} \le \Big(\frac{a-b}{a}\Big)^c \le \mathrm{e}^{-\frac{bc}{a}}.$$

Consequently, by (5) we get

$$\Pr(\bar{A}_Y) \le n^N \exp\Big(-\frac{x}{t}\sum_{\ell \in \mathcal{L}} y_\ell\Big).$$

Moreover, Fact 3 infers that $\sum_{\ell\in\mathcal{L}} y_\ell = e(Y,\mathcal{L}) \geq \frac{\alpha}{2}N^{\frac{4}{3}}$, and hence

$$\Pr(\bar{A}_Y) \leq n^N \exp\Big(-\frac{\alpha}{2}\frac{x}{t}N^{\frac{4}{3}}\Big).$$

Thus,

$$\begin{aligned}\Pr\big(\bigcup_Y \bar{A}_Y\big) &\leq \binom{N}{\lfloor\alpha N\rfloor} n^N \exp\Big(-\frac{\alpha}{2}\frac{x}{t}N^{\frac{4}{3}}\Big)\\ &\leq \Big(\frac{\mathrm{e}}{\alpha}\Big)^{\alpha N} n^N \exp\Big(-\frac{\alpha}{2}\frac{x}{t}N^{\frac{4}{3}}\Big)\\ &= \exp\bigg(N\Big(\alpha-\alpha\log\alpha+\log n-\frac{\alpha}{2}\frac{x}{t}\sqrt[3]{N}\Big)\bigg)\\ &\leq \exp\bigg(N\Big(\alpha-\alpha\log\alpha+\log n-\frac{\alpha}{2}\frac{3}{\alpha}\log n\frac{\sqrt[3]{N}}{t}\Big)\bigg), \qquad (8)\end{aligned}$$

where the last inequality follows from (3).

Since $\frac{\sqrt[3]{N}}{t} \sim 1$ we obtain that (8) tends to 0 as n goes to infinity. This yields that

$$\Pr\big(\bigcap_Y A_Y\big) > 0,$$

i.e. there is a graph H of order $N = q^3+q^2+q+1 = O(n^3\log^3 n)$ for which every subgraph of order $\lfloor\alpha N\rfloor$ contains G as an induced subgraph. In particular, for $\alpha = \frac{1}{r}$ we have $H \underset{ind}{\longrightarrow} (G)^v_r$ and $cl(H) = cl(G)$. This completes the proof of Theorem 1.

4 Concluding Remarks

With some additional work (see e.g. [3]) one can reduce the factor $n^3\log^3 n$ from Theorem 1 to $n^3\log n$.

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