Finding Folkman Numbers via MAX CUT Problem

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Abstract

In this note we report on our recent work, still in progress, regarding Folkman numbers. Let $f(2,3,4)$ denote the smallest integer n such that there exists a K_4–free graph of order n having that property that any 2–coloring of its edges yields at least one monochromatic triangle. It is well–known that such a number must exist [4,10]. For almost twenty years the best known upper bound, given by Spencer, was $f(2,3,4) < 3 \cdot 10^9$ [13]. Recently, the authors and Lu showed that $f(2,3,4) < 130\,000$ [2] and $f(2,3,4) < 10\,000$ [9]. However, it is commonly believed that, in fact, $f(2,3,4) < 100$. All previous bounds are based on an idea of Goodman [6]. It seems that such methods will not yield substantial further improvement. In this note we will generalize this idea by giving a necessary and sufficient condition for a graph G to yield a monochromatic triangle for every edge coloring. In particular, for any graph G we construct a graph H such that G is Folkman if and only if the value of the maximum cut of H is less than twice the number of triangles in G. We believe this technique may be used to find a new upper bound on $f(2,3,4)$.

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1 Introduction

Let r, k, l be positive integers with $k < l$, and let $\mathcal{F}(r, k, l)$ be a family of K_l–free graphs having the following property that if $G \in \mathcal{F}(r, k, l)$, then every r–coloring of edges of G must yield at least one monochromatic copy of K_k. J. Folkman showed in [4] that $\mathcal{F}(2, k, l) \neq \emptyset$. The general case, i.e. $\mathcal{F}(r, k, l) \neq \emptyset$, $r \geq 2$, was settled by J. Nešetřil and the second author in [10]. Let $f(r, k, l) = \min_{G \in \mathcal{F}(r,k,l)} |V(G)|$. The problem of determining the numbers $f(r, k, l)$ in general includes the classical Ramsey numbers and thus is not easy. In this note we focus to the case where $r = 2$ and $k = 3$. We will write $G \to (\triangle)$ and say that G arrows a triangle if every 2–coloring of G yields a monochromatic triangle. Since the Ramsey number $R(3, 3) = 6$ clearly $f(2, 3, l) = 6$, for $l > 6$. The value of $f(2, 3, 6) = 8$ was determined by R. Graham in [7], and $f(2, 3, 5) = 15$ by K. Piwakowski, S. Radziszowski and S. Urbański in [11]. In the remaining case, the upper bounds on $f(2, 3, 4)$ obtained from [4] and [10] are extremely large (iterated tower function). Consequently, in 1975, P. Erdös [3] offered \$100 for proving or disproving that $f(2, 3, 4) < 10^{10}$. Based on the idea of Goodman of counting triangles in a graph and in its complement [6] applied to random graphs P. Frankl and the second author came relatively close to the desired bound showing in [5] that $f(2, 3, 4) < 10^{12}$. Subsequently, J. Spencer in [13] refined this argument and proved $f(2, 3, 4) < 3 \cdot 10^9$ giving a positive answer to the question of Erdös [3]. Subsequently, F. Chung and R. Graham in [1] conjectured that $f(2, 3, 4) < 10^6$ and offered \$100 for a proof of disproof. Recently, L. Lu and independently the authors gave the computer assisted proof of $f(2, 3, 4) < 10\,000$ [9] and of $f(2, 3, 4) < 130\,000$ [2], respectively. Similarly as in [5] and [13] the proofs from [2] and [9] are based on the modification of the idea from Goodman's paper [6]. The idea explores the local property of every vertex neighborhood in a graph (See Corollary 2.2). While this property easily yields that a graph contains a monochromatic triangle in every edge coloring, it seems to be stronger than needed. We believe that this method may not yield substantial further improvement without additional modifications. We will give a necessary and sufficient condition for a graph G to yield a monochromatic triangle for every edge coloring. More precisely, for every graph G we will construct a weighted graph H such that G arrows a triangle if and only if the corresponding value to the maximum cut of H is less than twice number of triangles in G.

2 Counting blue and red triangles

In order to establish a necessary and sufficient condition for a graph G to yield a monochromatic triangle for every edge coloring, we will use a modification of an idea of [6]. For any blue–red coloring of G let $T_{BR}(v)$, $T_{BB}(v)$ and $T_{RR}(v)$ count the number of triangles containing vertex v, for which two edges incident to v are colored blue–red, blue–blue and red–red, respectively. The sum $\sum_{v\in V(G)} T_{BR}(v)$ counts 2 times the number of nonmonochromatic triangles. This is because each such triangle is counted once for two different vertices. On the other hand, the sum $\sum_{v\in V(G)} \big(T_{BB}(v)+T_{RR}(v)\big)$ counts 3 times the number of monochromatic triangles and once the number of nonmonochromatic triangles. Consequently, $G \rightarrow (\triangle)$ if and only if for every edge coloring of G the following holds

$$\sum_{v\in V(G)} T_{BR}(v) < 2 \sum_{v\in V(G)} \big(T_{BB}(v)+T_{RR}(v)\big). \tag{1}$$

Denote by $N(v)$ the set of neighbors of a vertex $v \in V$ and let $G[N(v)]$ be a subgraph of G induced on $N(v)$. Also we denote by $M(G)$ the size of the maximum cut of G. One can show using (1) the following Proposition.

Proposition 2.1 *Let $G=(V,E)$ be a graph that satisfies*

$$\sum_{v\in V(G)} M(G[N(v)]) < \frac{2}{3} \sum_{v\in V(G)} \big|E(G[N(v)])\big|. \tag{2}$$

Then, $G \rightarrow (\triangle)$.

A special case of Proposition 2.1 was used to determine the upper bounds on $f(2,3,4)$ in [2,5,9,13].

Corollary 2.2 (Frankl and Rödl [5]; Spencer [13]) *Let $G=(V,E)$ be a graph which satisfies*

$$M(G[N(v)]) < \frac{2}{3}\big|E(G[N(v)])\big| \tag{3}$$

for every vertex $v \in V(G)$. Then, $G \rightarrow (\triangle)$.

We extend the idea of Goodman [6] and give a necessary and sufficient condition for a graph G to yield a monochromatic triangle for every edge coloring. More precisely, for every graph $G=(V,E)$ with $t_\triangle = t_\triangle(G)$ triangles, we construct a weighted graph H with $2|E|$ vertices such that $G \rightarrow (\triangle)$ if and only if the value of the maximum cut of H is less than $2t_\triangle$.

Let G be a graph with the vertex set $V(G) = \{1, 2, ..., n\}$. For every vertex $i \in V(G)$, let G_i be a graph with $V(G_i) = \big\{(i,j) \mid j \in N(i)\big\}$ and $E(G_i) = \big\{\{(i,j),(i,k)\} \mid \{j,k\} \in E(G)\big\}$. Clearly G_i is isomorphic to the subgraph $G[N(i)]$ of G induced on the neighborhood $N(i)$. Now we define a weighted graph H as follows: $V(H) = \big\{(i,j) \in V(G) \times V(G) \mid (i,j) \in V(G_i)\big\}$ and $E(H) = E^+(H) \cup E^-(H)$, where $E^+(H) = \big\{\{(i,j),(i,k)\} \mid \{j,k\} \in G[N(i)]\big\}$ and $E^-(H) = \big\{\{(i,j),(j,i)\} \mid (i,j) \in V(G_i) \text{ and } (j,i) \in V(G_j)\big\}$. To every edge in E^+ and E^- we assign the weight 1 or $-\infty$, respectively. Clearly $|V(H)| = 2|E(G)|$, $|E^+(H)| = 3t_\triangle(G)$ and $|E^-(H)| = |E(G)|$. Note that the adjacency matrix of H is is a $2|E(G)| \times 2|E(G)|$ matrix with adjacency matrices of $G_i \cong G[N(i)]$ around the diagonal.

We say that H has a *positive cut* if the value of this cut is positive. Let $V(H) = V_1 \cup V_2$ be a positive cut. Since the value of each edge $\big\{(i,j),(j,i)\big\} \in E(H)$ is $-\infty$, we infer that $\big\{(i,j),(j,i)\big\} \in \binom{V_1}{2} \cup \binom{V_2}{2}$, whenever $\{i,j\} \in E(G)$. Consequently, each blue–red coloring of edges of G defines a bipartition of vertices of H and vice versa. Summarizing, the following holds.

Proposition 2.3 *There is a one to one correspondence between edge colorings of G and positive cuts of H.*

Based on Proposition 2.3 we proved the main result of this note, which we state here without the proof. Now $M(H)$ denotes the value of the maximum cut for a weighted graph H.

Theorem 2.4 *Let G be a graph. Then, $G \rightarrow (\triangle)$ if and only if $M(H) < 2t_\triangle(G)$.*

We will show how Theorem 2.4 can be used in the following simple example. Let G_{17} be a graph with the vertex set $V(G_{17}) = \{1, 2, ..., 17\}$ and the edge set defined as follows: $\{i < j\} \in E(G_{17})$ if $j - i$ is a quadratic residue of 17. One can check that G_{17} is K_4–free. Let G_{18} be a graph obtained from G_{17} by adding one additional vertex, say 18, connected to all vertices from $V(G_{17})$. Then, $|V(G_{18})| = 18$, $|E(G_{18})| = 85$, $t_\triangle(G_{18}) = 136$. Clearly G_{18} is K_5–free. In [8], R. Irving proved that $G \rightarrow (\triangle)$, thus establishing $f(2,3,5) \leq 18$. An alternative (computer assisted) proof of Irving's result is based on Theorem 2.4. Let H be the graph of order 170 from Theorem 2.4 that corresponds to G_{18}. Since $M(H) < 272 = 2 \cdot 136$,[3] Theorem 2.4 yields that $G_{18} \in \mathcal{F}(2,3,5)$. Note that we could not apply a simpler condition given by Proposition 2.1. This is because the maximum cut of $G_{18}[N(i)]$, $i = 1, ..., 17$, equals 14 and the maximum cut of $G_{18}[N(18)] \cong G_{17}$ equals 44. Hence, $\sum_{i \in V(G_{18})} M(G_{18}[N(i)]) =$

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[3] The authors used Biq Mac solver [12] to compute $M(H)$.

$17 \cdot 14 + 44 = 282$. Also, $\frac{2}{3}\sum_{i \in V(G_{18})} \left| E(G_{18}[N(i)]) \right| = \frac{2}{3}(17 \cdot 20 + 68) = 272$. We observe that due to the above equalities condition (2) fails and hence Proposition 2.1 cannot be applied.

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