A new proof of de Werra's theorem

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Abstract

A classical edge-coloring problem calls for a coloring of the edges of a given graph, using as few colors as possible, so that no two adjacent edges have the same color. Extending this problem de Werra proved the following result. Let $k \geq 2$ be an integer and $G = (V, E)$ be a graph. Then, there is a k-edge-coloring of G such that for every $v \in V$ and colors c_1, c_2 the number of edges adjacent to v that are colored with c_1 differs by at most 2 from the number of edges adjacent to v that are colored with c_2. Here we give another proof of this result in a somewhat more general setting using an algebraic approach.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A *k-edge-coloring* of G is a function $c : E \to \{1, \ldots, k\}$. Furthermore, c is called *proper* if no two adjacent edges have the same color. The smallest integer k for which a proper k-edge-coloring exists is called the *chromatic index* of G and is denoted by $\chi'(G)$. The well-known theorem of Vizing states that every graph G with maximum degree $\Delta(G)$ satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. We generalize the notion of determining of $\chi'(G)$ as follows.

Let $A(v)$ denote the set of edges adjacent to vertex $v \in V$. For an integer $b \geq 0$ call the k-edge-coloring $c : E \to \{1, \ldots, k\}$ *b-balanced* if for every vertex $v \in V$ and any two colors $1 \leq i, j \leq k$, $\big||A(v) \cap c^{-1}(i)| - |A(v) \cap c^{-1}(j)|\big| \leq b$. The reader can easily check that G has a 1-balanced $\Delta(G)$-edge-coloring if and only if $\chi'(G) = \Delta(G)$. Consequently, if G has no 1-balanced $\Delta(G)$-edge-coloring then by Vizing's theorem $\chi'(G) = \Delta(G) + 1$ and so G has a proper $(\Delta(G) + 1)$-edge-coloring. If we merge two colors in such a proper coloring, then we obtain a 2-balanced $\Delta(G)$-edge-coloring. Summarizing, we have just argued that every graph G has a 2-balanced k-edge-coloring for every $k \geq \Delta(G)$.

It is also easy to see that this is still true for $k = 2$. Indeed, if the graph is Eulerian then we can color its Eulerian tour alternately obtaining a 1-balanced (if the number of edges is even) or 2-balanced (if the number of edges is odd) 2-edge-coloring. Otherwise, if the graph is not Eulerian, we add one vertex and connect it to all vertices with odd degree, and then, we proceed as in the previous case.

Surprisingly, de Werra [7] showed that every graph has a 2-balanced edge-coloring for any number of colors (see also [5, 8, 9]).

Theorem 1.1 (de Werra [7]) *Let $k \geq 2$ be an integer. Then every graph has a 2-balanced k-edge-coloring.*

In this paper we give a different proof of Theorem 1.1 in a somewhat more general setting. First we introduce some notation.

A *hypergraph* $\mathcal{H}$ is a pair $(V, \mathcal{E})$, where V is a finite set of vertices, while $\mathcal{E} \subseteq 2^V$ is a set of edges. Clearly, every graph is a hypergraph with the size of every edge equal to 2. For each $v \in V$, let $deg(v)$ be the *degree* of v in H, namely the number of edges in $\mathcal{H}$ that contain v. We set $\Delta(\mathcal{H}) = \max_{v \in V(H)} deg(v)$ as the *maximum degree* of $\mathcal{H}$.

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and $k \geq 1$ be an integer. A *k-coloring* of V is a function $c : V \to \{1, \ldots, k\}$. Equivalently, it is a vertex partition $V = V_1 \cup \cdots \cup V_k$ (with $V_i = c^{-1}(i)$ for every $1 \leq i \leq k$). Let $b \geq 0$ be an integer. A k-coloring is *b-balanced* if for every edge $E \in \mathcal{E}$ and any two colors $1 \leq i, j \leq k$, $\big||E \cap V_i| - |E \cap V_j|\big| \leq b$. We prove:

Theorem 1.2 *Let $k \geq 2$ be an integer and $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $\Delta(\mathcal{H}) \leq 2$. Then, $\mathcal{H}$ has a 2-balanced k-coloring.*

Note that the above statement yields Theorem 1.1. Indeed, let $G = (V, E)$ and apply Theorem 1.2 to its *neighborhood-hypergraph* $\mathcal{H} = (E, \{A(v) : v \in V\})$ (clearly $\Delta(\mathcal{H}) = 2$, unless G is empty). Thus, $\mathcal{H}$ has a 2-balanced k-coloring which corresponds in G to the 2-balanced k-edge-coloring.

The main tool which we use to obtain Theorem 1.2 is the following lemma.

Lemma 1.3 *Let $0 < \varepsilon < 1$ and $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $\Delta(\mathcal{H}) \leq 2$. Then, there exists $U \subseteq V$ such that for every edge $E \in \mathcal{E}$,*

$$|E \cap U| \in \begin{cases} \{\varepsilon|E|, \varepsilon|E| + 1\} & \text{if } \varepsilon|E| \text{ is integer valued,} \\ \{\lfloor \varepsilon|E| \rfloor, \lceil \varepsilon|E| \rceil\} & \text{otherwise.} \end{cases}$$

We believe that this lemma is of independent interest and might have additional applications.

2 Proof of the key lemma

In this section we prove our main lemma. Its proof is based on a modification of the proof of the following Beck and Fiala's theorem [2].

Theorem 2.1 (Beck and Fiala [2]) *Every hypergraph $\mathcal{H}$ has a $(2\Delta(\mathcal{H}) - 1)$-balanced 2-coloring.*

There are generalizations of this result for an arbitrarily number of colors. See, e.g., Doerr and Srivastav [4] and Biedl et al. [3]. In this paper we are interested in a special case when $\Delta(\mathcal{H}) \leq 2$.

Proof of Lemma 1.3. Here we adapt the proof of the Beck and Fiala theorem presented by Alon and Spencer in [1].

Let $0 < \varepsilon < 1$ and $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $\Delta(\mathcal{H}) \leq 2$. To each vertex $v \in V$ we assign a variable x_v which will change as the proof progresses. Initially all $x_v = \varepsilon$. At the end all $x_v \in \{0, 1\}$ and U will be defined as

$$U = \{v \in V : x_v = 1\}.$$

We will have $0 \leq x_v \leq 1$ at all times and once $x_v \in \{0, 1\}$ it becomes final. At any time vertex v is called *fixed* if $x_v \in \{0, 1\}$; otherwise it is *floating*. For every edge $E \in \mathcal{E}$ we define its value as $\sum_{v \in E} x_v$. Hence, initially every edge has value $\varepsilon|E|$. Moreover, every edge with 0 or 1 floating vertices is called *safe*; otherwise it is called *active*.

We insist at all times that every active edge E has value $\varepsilon|E|$. This holds initially since $x_v = \varepsilon$ for all $v \in V$. Note that the number of active edges is never larger than the number of floating vertices. Indeed, let $V' \subseteq V$ and $\mathcal{E}' \subseteq \mathcal{E}$ be the set of floating vertices and active edges, respectively. Then, since every active edge contains at least 2 floating vertices and every vertex belongs to at most 2 edges we get

$$2|\mathcal{E}'| \leq |\{(v, E) : v \in E, v \in V', E \in \mathcal{E}'\}| \leq 2|V'|, \tag{1}$$

as required.

Consider every x_v as a variable if $v \in V'$ (i.e., v is floating); otherwise (if v is fixed) x_v is a constant. Hence,

$$\sum_{v \in E} x_v = \varepsilon|E| \text{ for every active edge } E \in \mathcal{E}'$$

is the system of $|\mathcal{E}'|$ linear equations. If $|\mathcal{E}'| < |V'|$, then the system is underdetermined (there are fewer equations than variables). Consequently, we can follow a line of solutions until reaching the boundary of the unit cube. That means at least one previously floating vertex has became fixed. This process does not touch fixed vertices and so safe edges stay safe (thought active edges may become safe).

Iterating the above procedure we eventually end up with $|\mathcal{E}'| = |V'|$. Then by (1) every active edge contains exactly 2 floating vertices. Now we round every variable corresponding to a floating vertex to the value 0 or 1 as follows. For every $v \in V'$,

$$x_v \text{ is set to } \begin{cases} 0 & \text{if } x_v < \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

We call this process as a *rounding phase*.

First assume that $\varepsilon|E|$ is integer valued. If E is an active edge with its floating vertices u and w, then $x_u + x_w = 1$. Hence, after rounding phase $x_u + x_w$ equals 1 or 2, and consequently,

$$\sum_{v \in E} x_v \in \{\varepsilon|E|, \varepsilon|E| + 1\}\,.$$

Note that in this case there are no safe edges with floating vertices. Hence, every safe edge satisfies

$$\sum_{v \in E} x_v = \varepsilon|E|.$$

Now assume that $\varepsilon|E|$ is not integer valued. After rounding phase every active edge satisfies

$$\varepsilon|E| - 1 < \sum_{v \in E} x_v \leq \varepsilon|E| + 1,$$

and hence,

$$\lceil \varepsilon|E| - 1 \rceil \leq \sum_{v \in E} x_v \leq \lfloor \varepsilon|E| + 1 \rfloor.$$

In particular, since $\varepsilon|E|$ is not an integer,

$$\lceil \varepsilon|E| - 1 \rceil = \lceil \varepsilon|E| \rceil - 1 = \lfloor \varepsilon|E| \rfloor$$

and

$$\lfloor \varepsilon|E| + 1 \rfloor = \lfloor \varepsilon|E| \rfloor + 1 = \lceil \varepsilon|E| \rceil.$$

Now consider safe edges. In this case when an edge becomes safe it contains exactly one floating vertex. Thus, the final value of every safe edge after rounding phase satisfies,

$$\lfloor \varepsilon|E| \rfloor \leq \sum_{v \in E} x_v \leq \lfloor \varepsilon|E| \rfloor + 1 = \lceil \varepsilon|E| \rceil.$$

which completes the proof. □

Corollary 2.2 *Let $0 < \varepsilon < 1$ and $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $\Delta(\mathcal{H}) \leq 2$. Then, there exists a partition of $V = U \cup W$ such that for every edge $E \in \mathcal{E}$,*

$$\varepsilon|E| - 1 < |E \cap U| \leq \varepsilon|E| + 1$$

and

$$(1 - \varepsilon)|E| - 1 \leq |E \cap W| < (1 - \varepsilon)|E| + 1.$$

In particular, for $\varepsilon = \frac{1}{2}$ Corollary 2.2 implies

$$-2 = \left(\tfrac{|E|}{2} - 1\right) - \left(\tfrac{|E|}{2} + 1\right) \leq |E \cap U| - |E \cap W| \leq \left(\tfrac{|E|}{2} + 1\right) - \left(\tfrac{|E|}{2} - 1\right) = 2,$$

and therefore, the following is also true.

Corollary 2.3 *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $\Delta(\mathcal{H}) \leq 2$. Then, $\mathcal{H}$ has a 2-balanced 2-coloring.*

3 2-balanced colorings

In this section we prove Theorem 1.2. We start with an easy auxiliary result.

Fact 3.1 *For an integer k let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph which has a 2-balanced k-coloring $V = V_1 \cup \cdots \cup V_k$. Then for every edge $E \in \mathcal{E}$, either*

$$\{|E \cap V_i| : 1 \le i \le k\} \subseteq \left\{ \left\lfloor \tfrac{|E|}{k} \right\rfloor - 1, \left\lfloor \tfrac{|E|}{k} \right\rfloor, \left\lceil \tfrac{|E|}{k} \right\rceil \right\}$$

or

$$\{|E \cap V_i| : 1 \le i \le k\} \subseteq \left\{ \left\lfloor \tfrac{|E|}{k} \right\rfloor, \left\lceil \tfrac{|E|}{k} \right\rceil, \left\lceil \tfrac{|E|}{k} \right\rceil + 1 \right\}.$$

Proof. Since the coloring is 2-balanced it is enough to prove that for every $1 \le i \le k$,

$$\left\lfloor \tfrac{|E|}{k} \right\rfloor - 1 \le |E \cap V_i| \le \left\lceil \tfrac{|E|}{k} \right\rceil + 1. \tag{3}$$

If for some $1 \le j \le k$, $|E \cap V_j| \le \lfloor \frac{|E|}{k} \rfloor - 2$, then since the coloring is 2-balanced $|E \cap V_i| \le \lfloor \frac{|E|}{k} \rfloor$ for every $1 \le i \le k$, and thus,

$$|E| = \sum_{i=1}^{k} |E \cap V_i| \le (k-1) \left\lfloor \tfrac{|E|}{k} \right\rfloor + \left(\left\lfloor \tfrac{|E|}{k} \right\rfloor - 2 \right) \le |E| - 2,$$

a contradiction. Similarly, we get a contradiction if for some $1 \le j \le k$, $|E \cap V_j| \ge \lceil \frac{|E|}{k} \rceil + 2$. Then, for every $1 \le i \le k$, $|E \cap V_i| \ge \lceil \frac{|E|}{k} \rceil$, and hence,

$$|E| = \sum_{i=1}^{k} |E \cap V_i| \ge (k-1) \left\lceil \tfrac{|E|}{k} \right\rceil + \left(\left\lceil \tfrac{|E|}{k} \right\rceil + 2 \right) \ge |E| + 2.$$

Consequently, (3) holds. □

Proof of Theorem 1.2. We proceed by induction on k. For $k = 2$ the statement follows from Corollary 2.3. Now let us assume that Theorem 1.2 is true for every $2 \le k' < k$. We show that it also holds for k.

First we apply Lemma 1.3 with $\varepsilon = \frac{1}{k}$. This gives a partition of vertices $V = V_1 \cup W$ such that for every $E \in \mathcal{E}$,

$$|E \cap V_1| \in \left\{ \left\lfloor \tfrac{|E|}{k} \right\rfloor, \left\lceil \tfrac{|E|}{k} \right\rceil, \tfrac{|E|}{k} + 1 \right\}. \tag{4}$$

By the inductive hypothesis applied to the hypergraph $\mathcal{H}[W] = (W, \{E \cap W : E \in \mathcal{E}\})$ with $k' = k - 1$, $\mathcal{H}[W]$ has a 2-balanced $(k-1)$-coloring $W = V_2 \cup \cdots \cup V_k$. It remains to show that $V = V_1 \cup \cdots \cup V_k$ is 2-balanced as well. In order to this end we show that for every edge $E \in \mathcal{E}$,

$$\max_{1 \le i \le k} |E \cap V_i| - \min_{1 \le i \le k} |E \cap V_i| \le 2. \tag{5}$$

To finish the proof it is enough to show that for every E satisfying (4), (5) holds.

First assume that $|E \cap V_1| \in \left\{\lfloor \frac{|E|}{k} \rfloor, \lceil \frac{|E|}{k} \rceil\right\}$. Let $|E| = km + r$, where $m \in \mathbb{Z}$ and $0 \leq r \leq k-1$. If $r = 0$ then $|E \cap V_1| = m$ and $|E \cap W| = (k-1)m$, and hence, $\lfloor \frac{|E \cap W|}{k-1} \rfloor = \lceil \frac{|E \cap W|}{k-1} \rceil = m$. Otherwise, if $1 \leq r \leq k-1$ then $|E \cap V_1| \in \{m, m+1\}$ and $|E \cap W| \in \{(k-1)m + r - 1, (k-1)m + r\}$, and hence, $\frac{|E \cap W|}{k-1} \in \{m + \frac{r-1}{k-1}, m + \frac{r}{k-1}\}$. Consequently, since $k \geq 3$, $m \leq \lfloor \frac{|E \cap W|}{k-1} \rfloor \leq \lceil \frac{|E \cap W|}{k-1} \rceil \leq m+1$ for any $0 \leq r \leq k-1$. Thus, by Fact 3.1 (applied to $\mathcal{H}[W]$ with the 2-balanced $(k-1)$-coloring $W = V_2 \cup \cdots \cup V_k$), either

$$\{|(E \cap W) \cap V_i| : 2 \leq i \leq k\} \subseteq \{m-1, m, m+1\}.$$

or

$$\{|(E \cap W) \cap V_i| : 2 \leq i \leq k\} \subseteq \{m, m+1, m+2\}.$$

However, since $|E \cap V_1| \in \{m, m+1\}$ and $(E \cap W) \cap V_i = E \cap V_i$ for $2 \leq i \leq k$, either

$$\{|E \cap V_i| : 1 \leq i \leq k\} \subseteq \{m-1, m, m+1\}.$$

or

$$\{|E \cap V_i| : 1 \leq i \leq k\} \subseteq \{m, m+1, m+2\},$$

i.e., (5) holds, as required.

It remains to show that (5) also holds for $|E \cap V_1| = \frac{|E|}{k} + 1$. Let $|E| = km$, where $m \in \mathbb{Z}$. Note that if $|E \cap V_1| = m+1$ then $|E \cap W| = (k-1)m - 1$, and hence since $k \geq 3$, $\lfloor \frac{|E \cap W|}{k-1} \rfloor = m-1$ and $\lceil \frac{|E \cap W|}{k-1} \rceil = m$. Thus, by Fact 3.1 (applied to $\mathcal{H}[W]$), either

$$\{|(E \cap W) \cap V_i| : 2 \leq i \leq k\} \subseteq \{m-2, m-1, m\} \tag{6}$$

or

$$\{|(E \cap W) \cap V_i| : 2 \leq i \leq k\} \subseteq \{m-1, m, m+1\}. \tag{7}$$

We show that the latter holds. Assume this is not the case. Then, by (6),

$$(k-1)m - 1 = |E \cap W| = \sum_{i=2}^{k} |E \cap V_i| \leq (m-2) + (k-2)m = (k-1)m - 2,$$

a contradiction. Consequently, (7) holds and since $|E \cap V_1| = m+1$,

$$|E \cap V_i| \in \{m-1, m, m+1\}$$

for every $1 \leq i \leq k$, as required.

This completes the proof of Theorem 1.2. □

4 1-balanced colorings

Here we find a sufficient condition for a hypergraph $\mathcal{H}$ with $\Delta(\mathcal{H}) \leq 2$ to have a 1-balanced k-coloring.

We start with a variation of Lemma 1.3. For a given hypergraph a *cycle* of length ℓ is a sequence of its ℓ edges $E_1 E_2 \dots E_\ell$ such that both $E_i \cap E_{i+1} \neq \emptyset$, $1 \leq i \leq \ell - 1$, and $E_\ell \cap E_1 \neq \emptyset$. An *odd-cycle* is a cycle with odd length.

Lemma 4.1 *Let $0 < \varepsilon < 1$ and $\mathcal{H} = (V, \mathcal{E})$ be an odd-cycle-free hypergraph with $\Delta(\mathcal{H}) \leq 2$. Then, there exists $U \subseteq V$ such that for every edge $E \in \mathcal{E}$,*

$$|E \cap U| \in \{\lfloor \varepsilon |E| \rfloor, \lceil \varepsilon |E| \rceil\}. \tag{8}$$

Proof. In order to prove this statement we slightly modify the proof of Lemma 1.3 by changing the rounding phase (cf. (2)). Recall that V' and $\mathcal{E}'$ are the set of floating vertices and active edges, respectively. We already concluded that if $|V'| = |\mathcal{E}'|$ then every active edge contains exactly 2 floating vertices (cf. (1)). Let F consists of all 2-element subsets $\{u, w\} \subseteq V'$ such that $u, w \in E$ for some $E \in \mathcal{E}'$. Observe that since $\Delta(\mathcal{H}) \leq 2$, $H = (V', F)$ is a multigraph of order at most 2. Moreover, by assumption $\mathcal{H}$ contains no odd-cycles, and hence, H has no odd-cycles. Thus, H is a bipartite graph. Let $V' = V_1' \cup V_2'$ be its vertex bipartition. For every $v \in V'$,

$$x_v \text{ is set to } \begin{cases} 1 & \text{if } v \in V_1', \\ 0 & \text{otherwise.} \end{cases}$$

Note that now every active edge E with $u, w \in V'$ satisfies $x_u + x_w = 1$.

Now we show that after the new rounding phase the subset of vertices

$$U = \{v \in V : x_v = 1\}.$$

satisfies (8).

First assume that $\varepsilon|E|$ is integer valued. Let E be an active edge with its floating vertices u and w. Then, $x_u + x_w = 1$ both before and after the rounding phase. Consequently, $\sum_{v \in E} x_v = \varepsilon|E|$ and since in this case there are no safe edges containing floating vertices, the statement holds.

Now assume that $\varepsilon|E|$ is not integer valued. Observe that after the new rounding phase every edge (both active and safe) satisfies

$$\lfloor \varepsilon |E| \rfloor \leq \sum_{v \in E} x_v \leq \lfloor \varepsilon |E| \rfloor + 1 = \lceil \varepsilon |E| \rceil,$$

as required. □

Theorem 4.2 *Let $k \geq 2$ be an integer and $\mathcal{H} = (V, \mathcal{E})$ be an odd-cycle-free hypergraph with $\Delta(\mathcal{H}) \leq 2$. Then, $\mathcal{H}$ has a 1-balanced k-coloring.*

Proof. The proof goes along the lines of the proof of Theorem 1.2. We proceed by induction on k. For $k = 2$ the statement follows from Lemma 4.1 applied with $\varepsilon = \frac{1}{2}$.

Now let us assume that Theorem 4.2 is true for every $2 \leq k' < k$. We show that it also holds for k.

First we apply Lemma 4.1 with $\varepsilon = \frac{1}{k}$. This gives a partition of vertices $V = V_1 \cup W$ such that for every $E \in \mathcal{E}$,

$$|E \cap V_1| \in \left\{ \left\lfloor \frac{|E|}{k} \right\rfloor, \left\lceil \frac{|E|}{k} \right\rceil \right\}. \tag{9}$$

By the inductive hypothesis applied to the odd-cycle-free hypergraph $\mathcal{H}[W] = (W, \{E \cap W : E \in \mathcal{E}\})$ with $k' = k - 1$, $\mathcal{H}[W]$ has a 1-balanced $(k-1)$-coloring $W = V_2 \cup \cdots \cup V_k$. It remains to show that $V = V_1 \cup \cdots \cup V_k$ is 1-balanced as well. In order to this end we show that for every edge $E \in \mathcal{E}$ and $2 \leq i \leq k$,

$$|E \cap V_i| \in \left\{ \left\lfloor \frac{|E|}{k} \right\rfloor, \left\lceil \frac{|E|}{k} \right\rceil \right\}.$$

Let $|E| = km + r$, where $m \in \mathbb{Z}$ and $0 \leq r \leq k-1$. If $r = 0$ then by (9) $|E \cap V_1| = \lfloor \frac{|E|}{k} \rfloor = \lceil \frac{|E|}{k} \rceil = m$. Hence, $|E \cap W| = (k-1)m$ and $\lfloor \frac{|E \cap W|}{k-1} \rfloor = \lceil \frac{|E \cap W|}{k-1} \rceil = m$, as required.

If $0 < r \leq k-1$ then by (9) $|E \cap V_1| \in \{m, m+1\}$, and hence, $|E \cap W| \in \{(k-1)m + r - 1, (k-1)m + r\}$. Thus, $\frac{|E \cap W|}{k-1} \in \{m + \frac{r-1}{k-1}, m + \frac{r}{k-1}\}$. Consequently, $m \leq \lfloor \frac{|E \cap W|}{k-1} \rfloor \leq \lceil \frac{|E \cap W|}{k-1} \rceil \leq m+1$. □

Let $G = (V, E)$ be a bipartite graph. Then its neighborhood-hypergraph is odd-cycle-free with maximum degree 2. Consequently, Theorem 4.2 yields the following result, which was previously proved by de Werra [7].

Theorem 4.3 (de Werra [7]) *Let $k \geq 2$ be an integer. Then every bipartite graph has a 1-balanced k-edge-coloring.*

As a matter fact, Theorem 4.3 yields Theorem 1.1. Let $G = (V, E)$ be a graph. We arbitrarily add an orientation to each edge obtaining a new graph $\overrightarrow{G} = (V, \overrightarrow{E})$. Replace in $\overrightarrow{G}$ every vertex v by v^- and v^+ and every oriented edge (u, v) by $\{u^-, v^+\}$. Let G' be the resulting bipartite graph. By Theorem 4.3 G' has a 1-balanced k-edge-coloring. Collapsing every vertex pair (v^-, v^+) back into a single vertex v, we turn this 1-balanced k-edge-coloring of G' into a 2-balanced k-edge-coloring of G.

It is also worth mentioning that Hilton and de Werra [5] determined a very nice sufficient condition for a graph to have a 1-balanced k-edge-coloring.

Theorem 4.4 (Hilton and de Werra [5]) *Let $k \geq 2$ be an integer and $G = (V, E)$ be a graph. If k does not divide $deg(v)$ for every $v \in V$, then G has a 1-balanced k-coloring.*

It was also noted in [5] that Theorem 4.4 yields Theorem 1.1. Indeed, let $G = (V, E)$ be a graph. Construct from G a new graph G' by adjoining a pendant edge to each vertex $v \in V$. Then apply Theorem 4.4 to G'. Clearly, the corresponding edge-coloring of G is 2-balanced.

5 Concluding remarks

In this paper we showed that every hypergraph $\mathcal{H}$ with $\Delta(\mathcal{H}) \leq 2$ has a 2-balanced k-coloring. In fact, by applying Lemma 1.3 repeatedly we can find such a coloring in polynomial time. However, it is NP-complete to determine for a fixed $k \geq 3$ whether $\mathcal{H}$ has a 1-balanced k-coloring. In fact, this immediately follows from a result of Holyer [6]. He showed that it is NP-complete to determine the chromatic index $\chi'(G)$ of an arbitrarily graph G. Recall that by Vizing's theorem, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. Let G be a graph with $\Delta \geq 3$. Set $k = \Delta(G)$. If the neighborhood-hypergraph $\mathcal{H}$ of G has a 1-balanced k-coloring, then the chromatic index of G is $\Delta(G)$. Otherwise, if $\mathcal{H}$ has no 1-balanced k-coloring, then the chromatic index of G is $\Delta(G) + 1$.

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