Vertex colorings of graphs without short odd cycles

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July 21, 2009

Abstract

Motivated by the work of Nešetřil and Rödl on "Partitions of vertices", we are interested in obtaining some quantitative extensions of their result. In particular, given a natural number r and a graph G of order m with odd girth g, we show the existence of a graph H with odd girth at least g and order that is poly-logarithmic in m such that every r-coloring of the vertices of H yields a monochromatic and induced copy of G.

1 Introduction

In 1976, Nešetřil and Rödl [7] proved the following result. Let $\mathcal{F}$ be a finite set of *forbidden* 2-connected graphs and r be a fixed natural number. Then for every graph G such that G does not contain a copy of any graph $F \in \mathcal{F}$, there exists a graph H that also does not contain a copy of any $F \in \mathcal{F}$ such that every r-coloring of the vertices of H yields a monochromatic and induced copy of G. In the proof, Nešetřil and Rödl consider a k-uniform hypergraph $\mathcal{H}$ with chromatic number greater than r (the chromatic number of $\mathcal{H}$ is the smallest number of colors needed to color the vertices of $\mathcal{H}$ so that there is no monochromatic hyperedge) such that $\mathcal{H}$ does not contain any cycle of length less than g. Their technique involves embedding the graph G into each hyperedge of $\mathcal{H}$ to obtain the graph H. Clearly, since the chromatic number of $\mathcal{H}$ is greater than r, every r-coloring of the vertices of H will yield a monochromatic hyperedge of $\mathcal{H}$, and hence a monochromatic and induced copy of G. It should be noted that their proof relies on the existence of such k-uniform hypergraphs which was proved by Erdős and Hajnal [4] using probabilistic methods.

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Lovász [6] gave a deterministic construction for these hypergraphs. In both cases, the order of $\mathcal{H}$, and so of H, is exponential in the order of G.

In this paper, we are interested in quantitative extensions of the Nešetřil-Rödl result in the particular case when G does not contain cycles of odd length less than g. Let the *odd girth* of a graph G be the length of the shortest odd cycle contained in G. We prove the following theorem.

Theorem 1 *Let $r \geq 1$ and $g \geq 3$, g odd, be natural numbers and let $G = (V, E)$ be a graph of order $m \geq 3$ and odd girth g. Let $n = \lceil (4rm \log m)^{2g+1} \rceil$. Then, there exists a graph H of order at most n and odd girth at least g such that every r-coloring of the vertices of H yields a monochromatic and induced copy of G.*

An asymptotic version of Theorem 1 is given below.

Corollary 2 *Let $r \geq 1$ and $g \geq 3$, g odd, be fixed natural numbers. Then, for every graph G of order m and odd girth g, there exists a graph H of order $O((m \log m)^{2g+1})$ and odd girth at least g such that every r-coloring of its vertices yields a monochromatic and induced copy of G.*

We supplement our main result with the specific case when G is a cycle of length g. We show the existence of a graph H of girth g such that every r-coloring will yield a monochromatic copy of a cycle of length g. In fact, every copy of a cycle of length g is induced. Note that we forbid all cycles of length less than g and do not require g to be odd in this case.

Theorem 3 *For given natural numbers $r \geq 1$ and $g \geq 3$ let $n = \lceil ((2r)^g (2g \cdot g!)^2)^{2(g-1)} \rceil$. Then, there exists a graph H of order at most n and girth g such that every r-coloring of H yields a monochromatic copy of a cycle of length g.*

More hypergraph notation and terminology: A *hypergraph* $\mathcal{G}$ is defined as a pair $(V, \mathcal{E})$, where V is a set of vertices, and $\mathcal{E} \subseteq 2^V$ is a set of hyperedges (edges, for convenience). The *order* of a hypergraph is the size of its vertex set. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{E} = \{E_1, E_2, \ldots, E_m\}$. If a vertex $v_i \in V$ is contained in edge E_j, then we say that v_i is *incident* to the edge E_j. The *incidence matrix* A of $\mathcal{G}$ is an $(n \times m)$ (0,1)-matrix such that $A(v_i, E_j) = 1$ if and only if v_i is incident to E_j. The number of edges containing a particular vertex v_i is the *degree* of v_i. We say that a hypergraph $\mathcal{G} = (V, \mathcal{E})$ is k-uniform if every edge $E \in \mathcal{E}$ has cardinality exactly k. It is k-regular if every vertex $v \in V$ is incident to exactly k edges. Observe that if $\mathcal{G} = (V, \mathcal{E})$ is a k-uniform k-regular hypergraph then $|V| = |\mathcal{E}|$. An alternating sequence of vertices and edges in $\mathcal{G}$ of the form $(v_1, E_1, v_2, E_2, \ldots, v_\ell, E_\ell, v_{\ell+1})$ is called a *tour* of length ℓ if $v_i \neq v_{i+1}$, $E_j \neq E_{j+1}$, and each E_i is incident to both v_i and v_{i+1} for any $i \in \{1, 2, \ldots, \ell\}$ and $j \in \{1, 2, \ldots, \ell - 1\}$. A tour $(v_1, E_1, v_2, E_2, \ldots, v_\ell, E_\ell, v_{\ell+1})$ is a *cycle* of length ℓ in $\mathcal{G}$ if $v_{\ell+1} = v_1$ and $v_i \neq v_j$ and $E_i \neq E_j$ for any $i, j \in \{1, 2, \ldots, \ell\}$ such that $i \neq j$. The *girth* of a hypergraph $\mathcal{G}$ is defined to be the length of the shortest cycle in $\mathcal{G}$ and consequently, the *odd girth* of a hypergraph is the length of its shortest odd cycle.

2 Existence of hypergraphs of given girth g

In order to prove Theorem 1, we need the following result. Given any integers $k \geq 2$ and $g \geq 2$, there exists a k-uniform k-regular hypergraph $\mathcal{G}$ with girth g. This was proved in the context of incidence structures by Payne and Tinsley [8] (see also Sauer [9] who proved a somewhat weaker result). For the sake of completeness and clarity, we present the result of our interest from [8] in the language of hypergraphs.

Theorem 4 (Payne and Tinsley [8]) *For integers $k \geq 2$ and $g \geq 2$, there exists a k-uniform k-regular hypergraph with girth g and order n satisfying*

$$n \leq \frac{(k-1)^{2g+1}-1}{k-2}. \tag{1}$$

Proof. We proceed by induction on g.

Base case ($g = 2$): For all $k \geq 2$, let $\mathcal{G} = (V, \mathcal{E})$ be the hypergraph whose incidence matrix is a $(k \times k)$ matrix consisting of all ones. Since each row consists of k ones, it implies that every vertex has degree exactly k. Every column has exactly k ones implying that every edge in $\mathcal{G}$ has size k. Thus, $\mathcal{G}$ is a k-uniform k-regular hypergraph with girth 2.

Inductive step ($g > 2$): This part of the proof is divided into two steps:

1. Given a k-uniform k-regular hypergraph $\mathcal{G}$ with girth $g-1$, we construct a k-uniform k-regular hypergraph $\mathcal{H}$ with girth at least g.

2. If $\mathcal{H}$ has girth greater than g, then we can obtain a k-uniform k-regular hypergraph $\mathcal{K}$ with girth exactly g.

Step 1 (construction): Let $\mathcal{G} = (V, \mathcal{E})$ be a k-uniform k-regular hypergraph of order n and girth $g-1$ with $V = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$. Let A be its incidence matrix. Since $\mathcal{G}$ is k-uniform and k-regular, A has exactly k ones in each row and each column with a total of nk ones. Define a $(q \times q)$ (0,1)-matrix Q as follows,

$$Q(i, i+1) = 1 \text{ for } i = 1, \ldots, q-1 \quad \text{and} \quad Q(q, 1) = 1.$$

With Q defined as above, it is easy to check that for any $0 \leq \ell \leq q$, we have

$$Q^\ell(i, i+\ell) = 1 \text{ for } i = 1, \ldots, q-\ell,$$
$$Q^\ell(i, i+\ell-q) = 1 \text{ for } i = q-\ell+1, \ldots, q,$$

and hence, Q^ℓ is an identity matrix (of order q) if and only if $\ell \equiv 0 \pmod{q}$. For an integer ℓ we say that Q^ℓ maps x to y if $Q^\ell(x, y) = 1$. Thus,

$$Q^\ell \text{ maps } x \text{ to itself if and only if } \ell \equiv 0 \pmod{q}. \tag{2}$$

Set $q = 2^{nk}$. We now construct matrix B from A as follows:

(i) Replace each 1 in A by a distinct power of the matrix Q as follows. If $A(v_i, E_j) = 1$, replace it with the matrix $Q_{v_i,E_j} = Q^{\ell}$, where $\ell \in \{1, 2, 2^2, \ldots, 2^{nk-1}\}$.

(ii) Replace each 0 in A by a zero matrix of order q.

The matrix B has exactly k ones in each row and k ones in each column and hence is the incidence matrix of some k-uniform k-regular hypergraph, say $\mathcal{H}$, of order nq. If $(w_1, F_1, w_2, F_2, \ldots, w_s, F_s, w_{s+1})$ with $w_{s+1} = w_1$ is a cycle in $\mathcal{H}$, we can trace a closed tour (not necessarily a cycle) of the same length in $\mathcal{G}$. Therefore, the girth of $\mathcal{H}$ is at least as large as the girth of $\mathcal{G}$, and hence, at least $g - 1$. However, we will show that the girth of $\mathcal{H}$ is strictly greater than $g - 1$.

Let us suppose that $C = (w_1, F_1, w_2, F_2, \ldots, F_{g-1}, w_g)$ with $w_g = w_1$ is a cycle of length $g - 1$ in $\mathcal{H}$. Note that every 1 in C comes from distinct Q_{v_j,E_k}; for otherwise, we can find a cycle of length less than $g - 1$ in $\mathcal{G}$, a contradiction. Also observe that the submatrix formed by restricting B to the rows $\{w_1, w_2, \ldots, w_{g-1}\}$ and columns $\{F_1, F_2, \ldots, F_{g-1}\}$ is a $(g - 1) \times (g - 1)$ (0,1)-matrix in which every row and every column has exactly two 1's. Let us assume that the 1 corresponding to (w_i, F_i) comes from Q_{v_i,E_i} and the 1 corresponding to (w_{i+1}, F_i) comes from Q_{v_{i+1},E_i} for each $1 \le i \le g - 1$. Furthermore, let the 1 corresponding to (w_1, F_1) be the 1 in the x-th row, $1 \le x \le q$, of Q_{v_1,E_1}. Since C is a cycle in $\mathcal{H}$, the matrix

$$Q_{v_1,E_1} Q^{-1}_{v_2,E_1} Q_{v_2,E_2} Q^{-1}_{v_3,E_2} \cdots Q_{v_{g-1},E_{g-1}} Q^{-1}_{v_1,E_{g-1}}$$

maps x to itself. Thus, by (2) and (i), we get

$$b_1 + b_2 2 + b_3 2^2 + \cdots + b_q 2^{nk-1} \equiv 0 \pmod{2^{nk}} \tag{3}$$

for some $b_i \in \{-1, 0, 1\}$. It is easy to observe that (3) cannot be satisfied for any non-trivial choice of b_i. This contradicts our supposition, and hence, the girth of $\mathcal{H}$ is at least g.

Step 2 (suppression): For a vertex-edge pair (v_i, E_j) with v_i incident to E_j we define its *girth* as the shortest cycle containing (v_i, E_j).

Suppose that the girth of $\mathcal{H}$ is greater than g. Then there is a vertex-edge pair, say (v_1, E_1), with girth greater than g as well. Let $B = (b_{ij})$ be the incidence matrix of $\mathcal{H}$. For convenience let $b_{11} = b_{12} = \cdots = b_{1k} = b_{21} = b_{31} = \cdots = b_{k1} = 1$ and (v_1, E_1), (v_1, E_2) and (v_2, E_1) belong to a cycle of length greater than g. Note that it is always possible to re-arrange the rows and columns of B so that this is true. We suppress (v_1, E_1) in the following manner. Delete the first row and first column of B. Since $g > 2$, the resulting matrix restricted to the first $(k-1)$ rows and $(k-1)$ columns is a zero matrix. Replace each diagonal entry of this submatrix by a 1. We consider a sequence of suppressions in which at each stage we suppress a pair (v_i, E_j), v_i incident to E_j, with girth greater than g. This process eventually ends in a matrix in which every pair (v_i, E_j) with v_i incident to E_j has girth g. It is easily observed that such a matrix is the incidence matrix of a k-uniform k-regular hypergraph, say $\mathcal{K}$, with girth g.

Upper bound for the order of $\mathcal{K}$: It remains to show that the order of $\mathcal{K}$ satisfies (1). Let us suppose $\mathcal{K}$ is of minimal order n. Suppose v_i and v_j are two vertices in $\mathcal{K}$ such that the length of the shortest tour $(v_i, E_{i1}, \ldots, E_{j1}, v_j)$ between them is greater than g. Let $I(v_i) = \{E_{i1}, E_{i2}, \ldots, E_{ik}\}$ and $I(v_j) = \{E_{j1}, E_{j2}, \ldots, E_{jk}\}$, where $I(v)$ is the set of edges incident to vertex v. Let us form a new hypergraph $\mathcal{K}'$ by removing v_i from E_{i1} and making it incident with v_j and similarly, removing v_j from E_{j1} and making it incident with v_i. Thus, in $\mathcal{K}'$, $I(v_i) = \{E'_{j1}, E_{i2}, \ldots, E_{ik}\}$ and $I(v_j) = \{E'_{i1}, E_{j2}, \ldots, E_{jk}\}$, where $E'_{j1} = (E_{j1} \setminus v_j) \cup v_i$ and $E'_{i1} = (E_{i1} \setminus v_i) \cup v_j$. The reader can easily check that the girth of $\mathcal{K}'$ is g and (v_i, E'_{j1}) is a vertex-edge pair not belonging to any cycle of length g (otherwise, it would contradict the length of the shortest tour between v_i and v_j in $\mathcal{K}$). So, $\mathcal{K}'$ is a k-uniform k-regular hypergraph with girth g that has a vertex-edge pair (v_i, E'_{j1}) not belonging to any cycle of length g. As described before, we can suppress such pair to obtain a k-uniform k-regular hypergraph of girth g and order less than n, contradicting the minimality of n. Thus, any two vertices of $\mathcal{K}$ are connected by a tour of length at most g. Starting with a vertex v_i and counting the number of vertices v_j connected to v_i by tours of length $1, 2, \ldots, g$, we have

$$n \leq 1+(k-1)k+(k-1)^3k+\cdots+(k-1)^{2g-1}k = 1+(k-1)+(k-1)^2+\cdots+(k-1)^{2g},$$

which evaluates to the expression in (1). □

3 Proof of Theorem 1

The main idea we employ in the proof of Theorem 1 is similar to an approach taken by the first author and Rödl in [1, 2].

Let $r \geq 1$ be a natural number and $G = (V, E)$ be a given graph with odd girth g and order $m \geq 3$. Let

$$n = \lceil (4rm \log m)^{2g+1} \rceil. \tag{4}$$

To prove Theorem 1, we will prove that there exists a graph $H = (W, F)$ of order at most n and odd girth at least g such that any subgraph of H induced by a set of $\lfloor \frac{|W|}{r} \rfloor$ vertices contains an induced copy of G. Fix k such that

$$3rm \log m \leq k \leq 4rm \log m. \tag{5}$$

From Theorem 4, we know there exists a k-uniform k-regular hypergraph $\mathcal{H} = (W, \mathcal{F})$ with girth g such that

$$|W| \leq \frac{(k-1)^{2g+1} - 1}{k-2} \leq (k-1)^{2g+1} \leq n,$$

where the last inequality follows from (4) and (5). We will construct a "random" graph H on $\mathcal{H}$. We randomly partition each hyperedge $K \in \mathcal{F}$ into $K = X_1 \cup \cdots \cup X_m$ with $x \leq |X_i| \leq x+1$. Let $k = xm + q$. Then, by (5),

$$2r \log m \leq x \leq 4r \log m. \tag{6}$$

Let $V = \{v_1, \ldots, v_m\}$ be the vertex set of G. For each $u \in X_i$ and $w \in X_j$, $\{u, w\}$ is an edge in H if and only if $\{v_i, v_j\}$ is an edge in G. Note that H is well-defined because every pair of vertices in W is contained in at most one hyperedge of $\mathcal{F}$. Clearly, in this construction, K does not contain any odd cycles of length less than g. Moreover, since $\mathcal{H}$ itself has girth g, H has odd girth at least g. We will show that there is a graph $H = (W, F)$ constructed in this manner having the following property. For every set $U \subseteq W, |U| = \lfloor \frac{|W|}{r} \rfloor$, the graph $H[U]$ contains G as an induced subgraph.

For a fixed $U \subseteq W$, with $|U| = \lfloor \frac{|W|}{r} \rfloor$, let A_U denote the event that $H[U]$ contains no induced copy of G. For every $K \in \mathcal{F}$, let B_K denote the event that there exists some X_i in the partition of $K = X_1 \cup \cdots \cup X_m$ such that $X_i \cap U = \emptyset$. Note that if A_U occurs, then all events B_K, $K \in \mathcal{F}$ must occur. Thus,

$$A_U \subseteq \bigcap_{K \in \mathcal{F}} B_K.$$

Since all events B_K are independent,

$$\Pr(A_U) \leq \prod_{K \in \mathcal{F}} \Pr(B_K). \tag{7}$$

We now obtain an upper bound on $\Pr(B_K)$. For a fixed hyperedge $K \in \mathcal{F}$, let $|U \cap K| = u_K$. Then for a fixed i, $1 \leq i \leq m$, the probability that $U \cap X_i = \emptyset$ is equal to the probability that for a fixed partition $K = X_1 \cup \cdots \cup X_m$, a randomly chosen subset T with $|T| = u_K$ satisfies $T \cap X_i = \emptyset$. Hence, since $x \leq |X_i|$,

$$\Pr(B_K) \leq m \frac{\binom{k-x}{u_K}}{\binom{k}{u_K}} \leq m \left(\frac{k-x}{k}\right)^{u_K} \leq m \exp\left\{-\frac{x u_K}{k}\right\}. \tag{8}$$

Consequently, from (7) and (8), we obtain

$$\Pr(A_U) \leq m^{|\mathcal{F}|} \prod_{K \in \mathcal{F}} \exp\left\{-\frac{x u_K}{k}\right\} = m^{|W|} \exp\left\{-\frac{x}{k} \sum_{K \in \mathcal{F}} u_K\right\}.$$

Since $\mathcal{H}$ is k-regular, for any $U \subseteq W$, $\sum_{K \in \mathcal{F}} |U \cap K| = |U|k$, hence yielding

$$\sum_{K \in \mathcal{F}} u_K = \lfloor |W|/r \rfloor k,$$

and consequently,
$$\Pr(A_U) \leq m^{|W|} \exp\{-x \lfloor |W|/r \rfloor\}.$$

We can bound the probability that there exists some $U \subseteq W$ with $|U| = \lfloor \frac{|W|}{r} \rfloor$ such that the graph induced by $H[U]$ does not induce a copy of G by bounding the probability of the union of the events A_U over all subsets $U \subseteq W$. Thereby, obtaining

$$\Pr\left(\bigcup_U A_U\right) \leq \binom{|W|}{\lfloor |W|/r \rfloor} m^{|W|} \exp\{-x \lfloor |W|/r \rfloor\}$$
$$\leq (\mathrm{e}r)^{\lfloor |W|/r \rfloor} m^{|W|} \exp\{-x \lfloor |W|/r \rfloor\} \leq \exp\{\lfloor |W|/r \rfloor (1 + \log r + r \log m - x)\}.$$

By (6), $1 + \log r + r \log m - x \leq 1 + \log r - r \log m$. Since $m \geq 3$, $1 + \log r - r \log m < 1 + \log r - r \leq 0$ implying that $\Pr(\bigcup_U A_U) < 1$. Thus, we can conclude that $\Pr(\bigcap_U \overline{A}_U) > 0$, i.e., there exists a graph $H = (W, F)$ of order at most n and odd girth at least g such that every r-coloring of its vertices yields a monochromatic and induced copy of G, thus completing the proof of Theorem 1.

Remark 5 It is possible to improve the bound on the order of the graph H in Theorem 1 in certain specific cases. A *generalized p-gon* of order (s, t), in the language of hypergraphs, is a $(s+1)$-uniform $(t+1)$-regular hypergraph with diameter $\lfloor \frac{p}{2} \rfloor$ and girth p. It is known that finite generalized 4-gons and 6-gons of order (s, s) exist when s is a prime power and generalized 8-gons of order (s, s^2) exist when $s = 2^{2r+1}$, where r is a natural number (see [10] for details). For $p \in \{4, 6\}$, the order of the hypergraph is given by $\frac{s^p - 1}{s - 1}$. Setting $g = p + 1$ for $p \in \{4, 6\}$, the value of n in Theorem 1 can be improved to $\lceil (4rm \log m)^{g-1} \rceil$. In the case when $g = 9$ an improvement is also possible (by applying generalized 8-gons).

4 Proof of Theorem 3

The main idea of the proof of Theorem 3 is based on the seminal paper of Erdős [3] in which he proved the existence of graphs with high girth and high chromatic number.

For natural numbers $r \geq 1$ and $g \geq 3$, let
$$n = \left\lceil ((2r)^g (2g \cdot g!)^2)^{2(g-1)} \right\rceil \tag{9}$$
and
$$\theta = \tfrac{1}{2}\left(\tfrac{1}{g} + \tfrac{1}{g-1}\right) = \tfrac{2g-1}{2g(g-1)}.$$

Clearly, $\frac{1}{g} < \theta < \frac{1}{g-1}$. We consider the random graph $G(n, p)$ with
$$p = n^{\theta - 1}, \tag{10}$$
and show that in $G(n, p)$:

(i) The probability that there are more than $\frac{n}{2}$ cycles of length less than g (referred to as short cycles) is smaller than 1/2.

(ii) The probability that there exists a subset of vertices of size $\lfloor \frac{n}{2r} \rfloor$ that does not contain a copy of a cycle of length g (denoted by C_g) is smaller than 1/2.

The above two properties yield that there exists a graph G of order n with at most $\frac{n}{2}$ short cycles such that every subset of vertices of size $\lfloor \frac{n}{2r} \rfloor$ contains a copy of C_g. We then obtain H from G by removing all the short cycles from G. We do this by deleting one vertex from each short cycle. It is easy to observe that in the newly formed graph H (of order at least $\frac{n}{2}$), every set of $\lfloor \frac{|V(H)|}{r} \rfloor$ vertices of H contains a copy of C_g and the girth of H is precisely g.

Number of short cycles: Here we show that the probability, that the number of short cycles in $G(n,p)$ is greater than $\frac{n}{2}$, is less than 1/2. Let X be the random variable counting the number of cycles of length less that g. Then the expected number of short cycles in $G(n,p)$ is given by

$$E[X] = \sum_{i=3}^{g-1} \binom{n}{i} \frac{(i-1)!}{2} p^i \leq \sum_{i=3}^{g-1} \frac{n^i p^i}{2i} \leq \frac{1}{6} \sum_{i=3}^{g-1} n^{\theta i} < \frac{g}{6} n^{\theta(g-1)}.$$

Thus, by Markov's inequality (see, e.g., inequality (1.3) in [5]),

$$\Pr\left(X \geq \frac{n}{2}\right) \leq \frac{E[X]}{n/2} \leq \frac{g}{3} n^{\theta(g-1)-1} = \frac{g}{3} n^{-\frac{1}{2g}},$$

and consequently, by (9)

$$\Pr\left(X \geq \frac{n}{2}\right) < \frac{1}{2}.$$

Containment of C_g: Now, we show that in $G(n,p)$, the probability that a subset of vertices of size $\lfloor \frac{n}{2r} \rfloor$ does not contain a cycle of length g is less that 1/2. Let A_U define the event that a fixed subset U with $|U| = \lfloor \frac{n}{2r} \rfloor$ does not contain a copy of C_g. We will use the following claim (which we shall prove subsequently).

Claim 1 *Let $m = \lfloor \frac{n}{2r} \rfloor$ and p be defined as in (10). Then,*

$$\Pr\left(G(m,p) \not\supseteq C_g\right) \leq \exp\left\{ -\frac{n^{\frac{2g-1}{2(g-1)}}}{(2r)^g (2g \cdot g!)^2} \right\}.$$

Assuming Claim 1, the probability that there exists some set of size $\lfloor \frac{n}{2r} \rfloor$ in G not containing a copy of C_g can be bounded from above by

$$\Pr\left(\bigcup_U A_U\right) \leq \binom{n}{m} \Pr\left(G(m,p) \not\supseteq C_g\right)$$
$$< 2^n \Pr\left(G(m,p) \not\supseteq C_g\right) = \exp\left\{ (\log 2)n - \frac{n^{\frac{2g-1}{2(g-1)}}}{(2r)^g (2g \cdot g!)^2} \right\} < \frac{1}{2},$$

where the last inequality follows from the value on n in (9). This completes the proof of Theorem 3.

It now remains to prove Claim 1.

Proof of Claim 1. Assume for simplicity that $\frac{n}{2r}$ is integer valued. Let A_α, $1 \le \alpha \le \binom{m}{g}\frac{(g-1)!}{2}$, range over the edge sets of possible copies of C_g and let B_α be the event that $G(m,p) \supseteq A_\alpha$. We apply Janson's inequality (see, e.g., Theorem 2.18 in [5]),

$$\Pr\left(\bigcap_\alpha \overline{B}_\alpha\right) \le \exp\left\{\frac{-\mu^2}{\mu+\Delta}\right\},$$

where

$$\mu = \sum_\alpha \Pr(B_\alpha) \qquad \text{and} \qquad \Delta = \sum_{\substack{A_\alpha \ne A_\beta \\ A_\alpha \cap A_\beta \ne \emptyset}} \Pr\left(B_\alpha \cap B_\beta\right).$$

We first obtain the following lower bound on μ,

$$\mu = \binom{m}{g}\frac{(g-1)!}{2}p^g > \frac{(m-g)^g}{2g}p^g \ge \frac{(mp)^g}{4g}, \tag{11}$$

since $(m-g)^g \ge m^g/2$ (as $m = \frac{n}{2r}$ and (9)). For the upper bound, we note that

$$\mu = \binom{m}{g}\frac{(g-1)!}{2}p^g < \frac{(mp)^g}{2g}. \tag{12}$$

Next, we bound Δ from above. Observe that C_g is a *strictly balanced graph*, i.e., for every proper subgraph $H \subsetneq C_g$,

$$\frac{|E(H)|}{|V(H)|} < \frac{|E(C_g)|}{|V(C_g)|} = 1. \tag{13}$$

In order to compute Δ, we split the sum according to the number of vertices in the intersection of A_α and A_β, $A_\alpha \ne A_\beta$. Suppose that A_α and A_β intersect in i vertices where $2 \le i \le g-1$. Clearly, $A_\alpha \cap A_\beta$ is a proper subgraph of C_g of order i. By (13),

$$|A_\alpha \cap A_\beta| < i,$$

and hence,

$$\Pr\left(B_\alpha \cap B_\beta\right) = p^{|A_\alpha \cup A_\beta|} = p^{2g-|A_\alpha \cap A_\beta|} \le p^{2g-(i-1)}. \tag{14}$$

We will also need the following inequality to compute Δ,

$$m^{2g-i}p^{2g-i+1} \le (mp)^g, \tag{15}$$

which is equivalent to $m^{g-i}p^{g-i+1} \leq 1$. The latter holds since for $m = \frac{n}{2r}$, $2 \leq i \leq g-1$, and $\theta < \frac{1}{g-1}$, we get

$$m^{g-i}p^{g-i+1} = \frac{n^{\theta(g-i+1)-1}}{(2r)^{g-i}} < n^{\theta(g-i+1)-1} \leq n^{\theta(g-1)-1} < 1,$$

as required.
Since A_α and A_β intersect in $2 \leq i \leq g-1$ vertices, we can evaluate Δ by summing over all i, the product: (number of ways to choose $2g-i$ vertices from the set of m vertices) $\times$ (the number of ways to choose g vertices for A_α) $\times$ (the number of ways to choose the i vertices of intersection) $\times$ (the number of cycles of length g on g vertices)^2 $\times$ $\Pr(B_\alpha \cap B_\beta)$. Hence, (14) yields

$$\begin{aligned}\Delta &\leq \sum_{2\leq i\leq g-1} \binom{m}{2g-i}\binom{2g-i}{g}\binom{g}{i}\left(\frac{(g-1)!}{2}\right)^2 p^{2g-(i-1)} \\ &\leq \sum_{2\leq i\leq g-1} \frac{m^{2g-i}}{(2g-i)!}\frac{(2g-i)!}{g!}g!\frac{(g-1)!^2}{4} p^{2g-i+1} = \frac{(g-1)!^2}{4}\sum_{2\leq i\leq g-1} m^{2g-i}p^{2g-i+1},\end{aligned}$$

and consequently, by (15),

$$\Delta \leq \frac{(g-1)!^2}{4}\sum_{2\leq i\leq g-1}(mp)^g < \frac{(g-1)!^2}{4}g(mp)^g = \frac{(g-1)!g!}{4}(mp)^g. \tag{16}$$

Using (11), (12) and (16), we bound the exponential term $\frac{-\mu^2}{\mu+\Delta}$ in Janson's inequality to obtain

$$\frac{-\mu^2}{\mu+\Delta} \leq \frac{-(mp)^{2g}}{(4g)^2\left[(mp)^g\left(\frac{1}{2g}+\frac{(g-1)!g!}{4}\right)\right]} \leq \frac{-(mp)^{2g}}{(4g)^2(mp)^g\left(\frac{g!^2}{4}\right)} \leq \frac{-(mp)^g}{(2g\cdot g!)^2},$$

and thus,

$$\Pr\left(G(m,p) \not\supseteq C_g\right) \leq \exp\left\{-\frac{(mp)^g}{(2g\cdot g!)^2}\right\} = \exp\left\{-\frac{n^{\frac{2g-1}{2(g-1)}}}{(2r)^g(2g\cdot g!)^2}\right\}.$$

The last equality follows from the choice of m and p. □

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