On k–Chromatically Connected Graphs

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Abstract

A graph G is chromatically k–connected if every vertex cutset induces a subgraph with chromatic number at least k. Thus, in particular each neighborhood has to induce a k–chromatic subgraph. In [3], Godsil, Nowakowski and Nešetřil asked whether there exists a k–chromatically connected graph such that every minimal cutset induces a subgraph with no triangles. We show that the answer is positive in a special case, when *minimal* is replaced by *minimum* cutsets. We will also answer another related question suggested by J. Nešetřil by proving the existence of highly chromatically connected graphs in which every vertex neighborhood induces a subgraph with a given girth.

Key words: chromatic connectivity, Kneser graphs

1 Introduction

We say that a graph $G = (V(G), E(G))$ is *chromatically k-connected* if every vertex cutset induces a subgraph with chromatic number at least k. Consequently, for all $v \in V(G)$ the neighborhood of v, denoted by $N_G(v)$, induces a k–chromatic subgraph. The concept of chromatically k-connected graphs likely first arose in Ramsey Theory. In [10], Nešetřil and the third author investigated which classes $\mathcal{C}$ of graphs have the edge-partition-property. The class $\mathcal{C}$ of graphs has the *edge-partition-property* if for every r and graph $G \in \mathcal{C}$ there exists a graph $H \in \mathcal{C}$ such that every r–coloring of edges of H yields a monochromatic copy of G. Extending earlier results of Folkman, in [10], it was proved that the class $\mathcal{A}$ of chromatically 3–connected graphs has the edge-partition-property.

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In 1987, Godsil, Nowakowski, and Nešetřil [3] further examined the chromatic connectivity of graphs. In particular, they proved that the chromatic connectivity of the Kneser graph $K(4n-1, n)$, which is K_4-free, is at least $\lceil n/6 \rceil + 1$. This led the authors of [3] to ask whether there is a k–chromatically connected graph with every minimal cutset inducing a subgraph with no triangles. We answer this question affirmatively in a special case when *minimal* cutsets are replaced by *minimum* cutsets.

Theorem 1 *For any $n \geq 6k$, the Kneser graph $K(4n-1, n)$ is k–chromatically connected and every minimum cutset induces a subgraph with no triangles.*

Furthermore, we will address another question raised by J. Nešetřil and show that for any $k \geq 1$, there exist k-chromatically connected graphs such that every vertex neighborhood induces a graph with a given girth. The *girth* of G, denoted by $g(G)$, is the length of the shortest cycle in G.

Theorem 2 *For any $k \geq 1$ and $l \geq 3$, there exists a k–chromatically connected graph such that every vertex neighborhood induces a graph with girth at least l.*

2 Proof of Theorem 1

For $n \geq 1$ and $m > 2n$, the Kneser graph $K(m, n)$ [6], is a graph with the set of vertices consisting of all n element subsets of an m element set, i.e. it is equal to $\binom{[m]}{n}$. Two vertices are adjacent if and only if the corresponding n–subsets are disjoint. The two propositions below relate to vertex connectivity of $K(m, n)$. For a given graph G the *vertex connectivity* of G is the minimum number of vertices that need to be deleted to make G disconnected. We denote this parameter by $\kappa(G)$. Proposition 3 states that the vertex connectivity of $K(m, n)$ is as large as possible, i.e. it is equal to the cardinality of every vertex neighborhood. Moreover, Proposition 4 states that every minimum vertex cutset is isomorphic to some vertex neighborhood, denoted by $N(v)$. Both propositions follow from a more general result of Tindell, which describes vertex connectivity of edge transitive graphs (See Proposition 2.4.2 and Corollary 2.4.6 in [11]).

Proposition 3 *The vertex connectivity of the Kneser graph K(m,n) is precisely d, i.e.*

$$\kappa(K(m, n)) = d,$$

where $d = \binom{m-n}{n}$.

Proposition 4 *Every minimum vertex cutset $\mathcal{K}$ of the Kneser graph K(m,n) satisfies $\mathcal{K} = N(v)$, for some $v \in V(K(m, n))$.*

Proof of Theorem 1. By [3] we get that $K(4n-1,n)$ is $(\lceil n/6\rceil+1)$–chromatically connected. Since $n \geq 6k$, $K(4n-1,n)$ is therefore k–chromatically connected. Proposition 4 implies that every minimum vertex cutset is induced by some vertex neighborhood. For every $v \in V(K(4n-1,n))$, its neighborhood $N(v)$ is $\triangle$–free because it induces a copy of $K(3n-1,n)$.

□

3 Proof of Theorem 2

3.1 Probabilistic construction

Before we can prove Theorem 2 we need some additional results.

Proposition 5 *For every $l \geq 3$ and $0 < \varepsilon < \frac{1}{2l}$ there exists $N = N(l,\varepsilon)$ such that for every $n \geq N$ there is a graph F on n vertices with the following properties:*

(i) $|deg_F(v) - n^\varepsilon| \leq \lfloor \frac{1}{12} n^\varepsilon \rfloor$, *for every* $v \in V(F)$;
(ii) $e(X, V(G)\backslash X) > |X|(\frac{n^\varepsilon}{3} - 1)$, *for every* $X \subset V(F)$ *with* $1 \leq |X| \leq \frac{n}{2}$;
(iii) $g(F) \geq l$.

Proof. Consider a random graph $G \sim G(n,p)$ with $p = n^{\varepsilon-1}$. Let:

- $\boldsymbol{A_1}$ be the event that there exists $v \in V(G)$ such that $|deg_G(v) - n^\varepsilon| \geq \lfloor \frac{1}{12} n^\varepsilon \rfloor$;
- $\boldsymbol{A_2}$ be the event that there exists $X \subset V(G)$ with $1 \leq |X| \leq \frac{n}{2}$ s.t. $e(X, V(G)\backslash X) \leq |X| \frac{n^\varepsilon}{3}$;
- $\boldsymbol{A_3}$ be the event that there exists a vertex $v \in V(G)$, which belongs to at least two cycles of length less than l.

We will show that $\boldsymbol{Pr}[\boldsymbol{A_i}] = o(1)$ for $i = 1,2,3$.

The cases for $i = 1,2$ follow from the standard application of Chernoff's inequality (See e.g. Theorem 8.5.2 in [1]).

Now we prove that $\boldsymbol{Pr}[\boldsymbol{A_3}] = o(1)$. Let $\boldsymbol{B_{i,j,k}}$ be the event that G contains cycles C_i and C_j of length i and j, respectively, which share exactly k consecutive vertices, where $3 \leq i,j \leq l$ and $1 \leq k \leq \min\{i,j\} - 1$. Then,

$$\boldsymbol{Pr}[\boldsymbol{B_{i,j,k}}] \leq \binom{n}{i+j-k} \cdot (i+j-k)! \cdot p^{i+j-k+1} = o(1),$$

since by assumption $\varepsilon \leq 1/2l$, and consequently

$$\boldsymbol{Pr}[\boldsymbol{A_3}] = \boldsymbol{Pr}[\text{There are } i,j,k \text{ s.t. } \boldsymbol{B_{i,j,k}} \text{ holds}] = o(1).$$

Let n be sufficiently large so that each of these three events $\boldsymbol{A_i}$'s have probability less than 1/3. Then,

$$\boldsymbol{Pr}[\boldsymbol{A_1^c} \cap \boldsymbol{A_2^c} \cap \boldsymbol{A_3^c}] > 0,$$

and consequently the following statement is true:

For every $l \geq 3$ and $0 < \varepsilon < \frac{1}{2l}$ there exists N such that for every $n \geq N$ there is a graph G on n vertices such that:

- $|deg_G(v) - n^{\varepsilon}| < \lfloor \frac{1}{12} n^{\varepsilon} \rfloor$, *for every* $v \in V(G)$;
- $e(X, V(G) \backslash X) > |X| \frac{n^{\varepsilon}}{3}$, *for every* $X \subset V(G)$ *with* $1 \leq |X| \leq \frac{n}{2}$;
- v *belongs to at most one cycle of length less than* l, *for every* $v \in V(G)$.

Consider such a graph $G \in G(n,p)$ and remove from each cycle of length less than l one edge. This way we obtain a new graph F with $g(F) \geq l$ which satisfies (iii). Since every vertex v belongs to at most one cycle of length less than l, we infer that

$$deg_G(v) - deg_F(v) = 0 \text{ or } 1.$$

This implies the first two conditions (i),(ii) and consequently F satisfies the statement of the Proposition 5.

□

The next Proposition is a special case of a well-known result of Erdös [2].

Proposition 6 *For every $k \geq 1$ and $l \geq 3$ there exists $M = M(k,l)$ such that for every $m \geq M$ there is a graph H on m vertices such that the following conditions hold:*

(i) *For every $X \subset V(H)$ with $|X| \geq \frac{1}{12} m$ the graph $H[X]$ induced on X satisfies $\chi(H[X]) \geq k$;*
(ii) $g(H) \geq l$.

Proof. We follow the lines of the proof of the well-known theorem of Erdös on graph with high girth and high chromatic number [2]. Indeed, let $0 < \delta < 1/l$. By [2] there exists M such that for any $m \geq M$ there is a graph H on m vertices which satisfies:

- $\alpha(H) \leq 3m^{1-\delta} \log m$;
- $g(H) \geq l$.

Let m be large enough to satisfy $\frac{1}{12}m/3m^{1-\delta}\log m \geq k$. Then, for every X with $|X| \geq \frac{1}{12}m$ we have

$$\begin{aligned}\chi(H[X]) &\geq |X|/\alpha(H[X]) \\ &\geq |X|/\alpha(H) \\ &\geq \frac{1}{12}m/3m^{1-\delta}\log m \\ &\geq k.\end{aligned}$$

□

Now we are ready to prove Theorem 2.

Proof of Theorem 2. For given $k, l \geq 3$, let H be a graph on m vertices obtained from Proposition 6 for $m \geq M = M(k, l)$. Similarly, let F be a graph on n vertices given by Proposition 5 for $n \geq N = N(6, \varepsilon)$, where $\varepsilon = 1/24$, i.e. $g(F) \geq 6$. Pick n sufficiently large such that $n \geq N$ and $\lceil \frac{5}{6}n^{\varepsilon} \rceil \geq M$, and set $m = \lceil \frac{5}{6}n^{\varepsilon} \rceil$. So, we obtain two graphs F and H on n and m vertices, which satisfy conditions of Propositions 5 and 6, respectively. Note, that since $C_3 \nsubseteq F$, the neighborhood of each vertex in $V(F)$ is an independent set. Similarly, since $C_4 \nsubseteq F$, two distinct vertices may share at most one neighborhood. Also in view of (i) of Proposition 5 $deg_F(v) > |V(H)|$, for every $v \in V(F)$. Now, for each vertex v of F consider a copy H_v of H embedded into its neighborhood $N_F(v)$. This way we obtain a new graph Γ with

$$V(\Gamma) = V(F)$$

and

$$E(\Gamma) = E(F) \cup \bigcup_{v \in V(F)} E(H_v).$$

We will prove that Γ satisfies the statement of the theorem.

First we show that Γ is k–chromatically connected. Let K be a vertex cut in Γ. Then K is also a vertex cut in F, since $F \subset \Gamma$. Hence,

$$F[V(F) \setminus K] = C_1 \sqcup ... \sqcup C_r,$$

for some r, i.e. K decomposes F into r disjoint connected components. We may assume that $|C_1| \leq n/2$. By (ii) of Proposition 5 (for graph F)

$$e(C_1, K) = e(C_1, V(F) \setminus C_1) > |C_1|(n^{\varepsilon}/3 - 1).$$

That means, there exists $v \in C_1$ s.t. $|N_F(v) \cap K| \geq n^{\varepsilon}/3$, which implies that in Γ the graph induced by $N_{\Gamma}(v) \cap K$ intersects a large portion of H_v. Indeed, since $deg_F(v) \leq \frac{13}{12}n^{\varepsilon}$ we get $|N_F(v) - V(H_v)| \leq \frac{13}{12}n^{\varepsilon} - \lceil \frac{5}{6}n^{\varepsilon} \rceil \leq \frac{1}{4}n^{\varepsilon}$.

Consequently, any subset of $N_F(v)$ of size at least $n^\varepsilon/3$ must overlap H_v on at least $\frac{1}{12}n^\varepsilon$ vertices. But in view of (i) of Proposition 6, any subgraph of H_v, which is induced by a set of size at least $\frac{1}{12}n^\varepsilon$, must have a chromatic number at least k. Thus,

$$\chi(\Gamma[K]) \geq \chi(\Gamma[N_F(v) \cap K]) \geq \chi(H_v[V(H_v) \cap K]) \geq k,$$

i.e. Γ is k–chromatically connected.

To show that for every vertex v we have $g(\Gamma[N_\Gamma(v)]) \geq l$ we need to analyse the structure of $N_\Gamma(v)$. Obviously $N_F(v) \subseteq N_\Gamma(v)$. Also for every $u \in N_F(v)$ the neighborhood of u in F contains v. Moreover, since $C_4 \nsubseteq F$, we conclude that for all $u \neq w \in N_F(v)$ we have $N_F(u) \cap N_F(w) = \{v\}$. Hence,

$$N_\Gamma(v) = N_F(v) \sqcup \bigsqcup_{u \in N_F(v)} N_u,$$

where $N_u \subseteq N_F(u) \setminus \{v\}$ for $u \in N_F(v)$. Finally, note that since $C_5 \nsubseteq F$, there is no edges between N_u's sets. Thus,

$$g(\Gamma[N_\Gamma(v)]) = g(\Gamma[N_F(v)]) = g(H_v) \geq l,$$

which completes the proof.

□

Observing that *girth* $\geq l$ is a decreasing property and $\chi \geq k$ is an increasing property, one can extend the argument from the above proof. Since every decreasing property $\mathcal{A}$ is given by forbidding a family of graphs $\mathcal{F}$, i.e. $\mathcal{A} = Forb(\mathcal{F})$, one can generalize Theorem 2 as follows:

Proposition 7 *Let $\mathcal{F}$ be a finite family of 2–connected graphs, and let $\mathcal{A} = Forb(\mathcal{F})$. Let $\mathcal{B}$ be an arbitrary increasing property such that $\mathcal{A} \cap \mathcal{B} \neq \emptyset$. Then, there exists a graph Γ such that every vertex neighborhood induces a graph with property $\mathcal{A}$ and every vertex cutset has property $\mathcal{B}$.*

Proof of the above proposition follows same lines as that of Theorem 2. The only minor change is in construction of graph H. One can fix $H^* \in \mathcal{A} \cap \mathcal{B}$ and construct a $|V(H^*)|$–uniform hypergraph $\mathcal{H}$ with the property that every small subset of $V(\mathcal{H})$ contains an edge of $\mathcal{H}$. The graph H can be constructed by inserting a copy of H^* into each edge of $\mathcal{H}$.

3.2 Constructive approach

In Section 3.1, we proved the existence of k–chromatically connected graph with a given girth in every vertex neighborhood by using the probabilistic methods. It is worth mentioning that there is a deterministic construction of graphs mentioned in Propositions 5 and 6. Indeed, Lubotzky, Phillips and Sarnak [8] and later Morgenstern [9] constructed an infinite family of regular Ramanujan graphs with arbitrarily high girth and chromatic number. Also note that a d–regular Ramanujan graph satisfies the condition (ii) of Proposition 5 for sufficiently large d, since the absolute value of the second largest eigenvalue of its adjacency matrix has the order $O(\sqrt{d}) = o(d)$, which implies (ii) (See e.g. [1]).

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