Expansion and lack thereof in randomly perturbed graphs

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March 20, 2007

Outline

1. Introduction
   - Random Graphs
   - Randomly perturbed graphs

2. Expansion
   - Expansion in the real world
   - Expansion in randomly perturbed graphs

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- Started out as pure math
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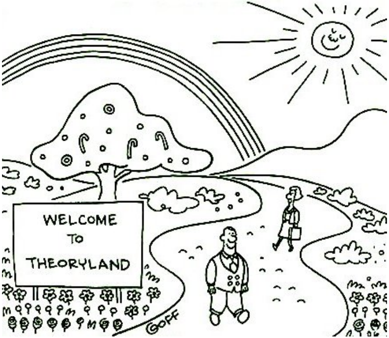

Experiments

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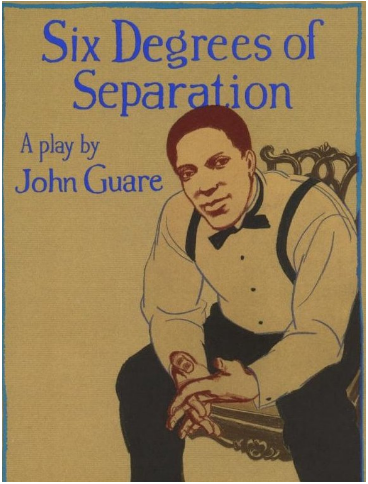

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Randomly perturbed graphs

Randomly perturbed graphs

Start with a pretty arbitrary graph $\overline{G}$, and perturb it by adding sparse random graph R, to obtain

$$G = \overline{G} + R.$$

Randomly perturbed graphs

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Based on

- Smoothed analysis [Spielman and Teng]

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- Diameter of a cycle plus a random matching
  [Bollobás and Chung]
- How many random edges make a dense graph Hamiltonian?
  [Bohman, Frieze, and Martin]

A proposed approach for real-world graphs

Theorems that hold for

a sufficiently arbitrary graph
and a sufficiently small perturbation

should be
valid predictions for real-world networks.

Example

Theorem

Let $\overline{G}$ be any connected n-graph, and let $R \sim \mathbb{G}_{n,\epsilon/n}$. Then, with high probability, $G = \overline{G} + R$ has

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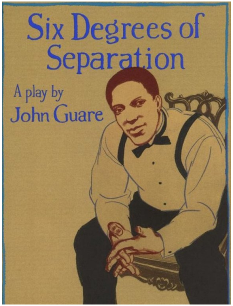

A scientific question

Is the randomly perturbed graph a good model for the real world?

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- Eigenvalue gap: For matrix M given by

$$M_{i,j} = \begin{cases} \deg(i), & \text{if } i = j; \\ -1, & \text{if } \{i, j\} \in E; \\ 0, & \text{otherwise}; \end{cases} \qquad \lambda_1(M) \geq \epsilon$$

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Good to have expansion and good not to have expansion, too.

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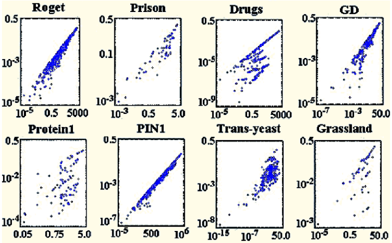
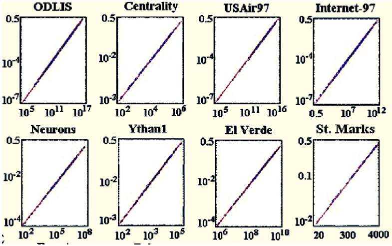
E. Estrada, Spectral scaling and good expansion properties in complex networks, *Europhysics Letters*, 73 (4), pp. 649–655 (2006).





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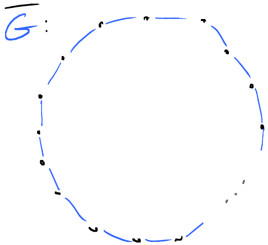
If $R \sim \mathbb{G}_{1\text{-}out}$ then G is an expander **whp**.

Proof if $R \sim \mathbb{G}_{n,\epsilon/n}$ then G not necess. expander



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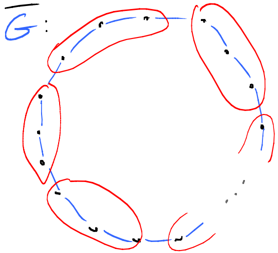

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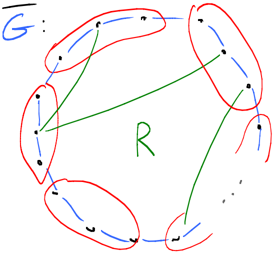

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This doesn't work unless k is a large enough constant. (And it shouldn't, since it's not true for $k = 1$.)

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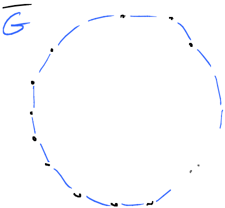
What if $\overline{G}$ is a cycle?



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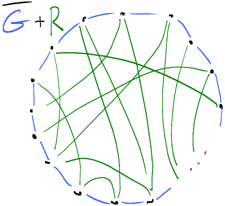

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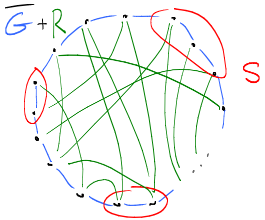

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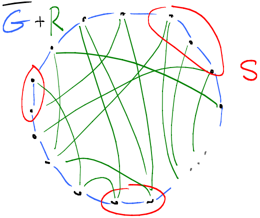

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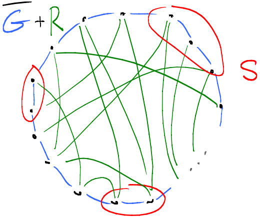

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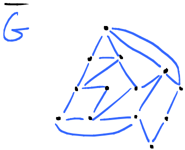
For general $\overline{G}$, need something to take the place of the cycle.



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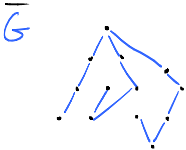

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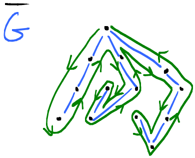

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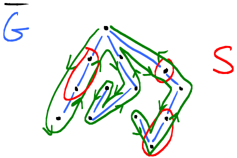

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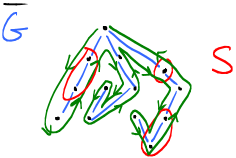

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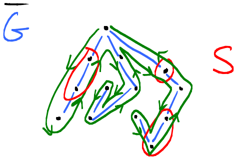

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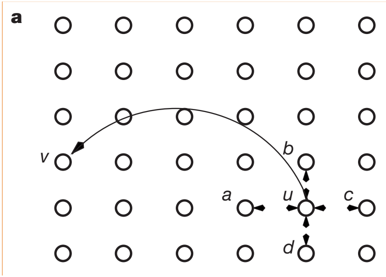
Curious extension of these techniques

Kleinberg's extension of Watts-Strogatz model

Figure 1 The navigability of small-world networks. **a,** The network model is derived from an $n \times n$ lattice. Each node, *u*, has a short-range connection to its nearest neighbours (*a, b, c* and *d*) and a long-range connection to a randomly chosen node, where node *v* is selected with probability proportional to $r^{-\alpha}$, where *r* is the lattice ('Manhattan') distance between *u* and *v*, and $\alpha \geqslant 0$ is a fixed clustering exponent. More generally, for $p,q \geqslant 1$, each node *u* has a short-range connection to all nodes within *p* lattice steps, and *q* long-range connections generated independently from a distribution with clustering exponent α. **b,** Lower bound from my charac-



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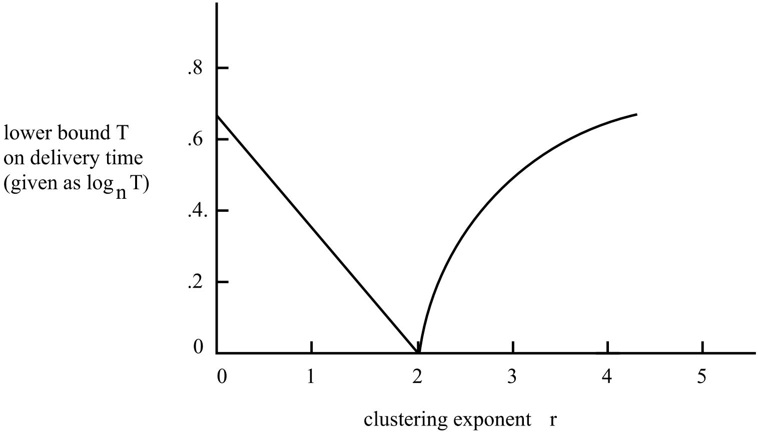

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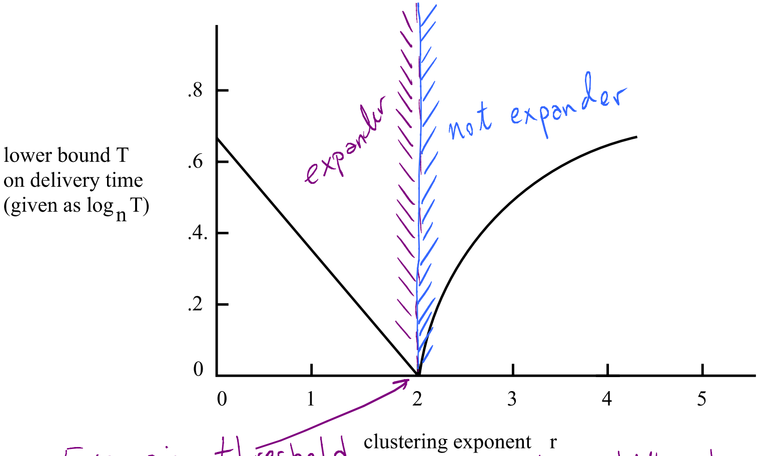

Expansion threshold
matches "algorithmic small world" value


ENTRANCE
To
MYSTERY SPOT

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