A Geometric Preferential Attachment Model of Networks II

Abraham D. Flaxman,
Microsoft Research

Alan M. Frieze,
Carnegie Mellon University

Juan Vera
University of Waterloo

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Outline

- Introduction
  - Preferential Attachment and its relatives
- Model
  - Geometric Preferential Attachment I
  - Geometric Preferential Attachment II
- Results
  - Theorems
  - Proof techniques
- Conclusion

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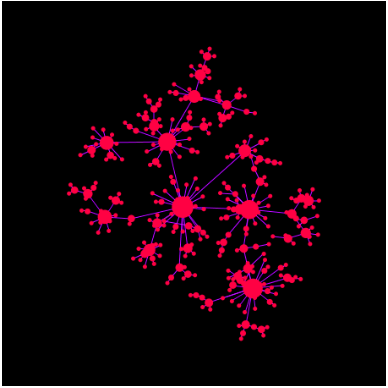

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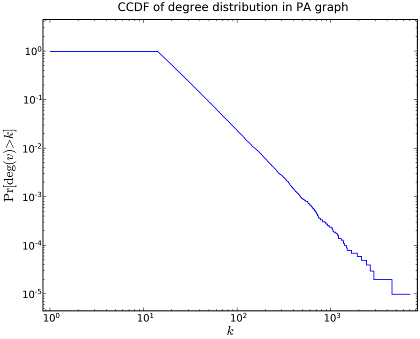

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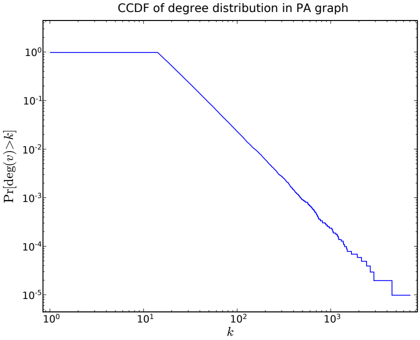

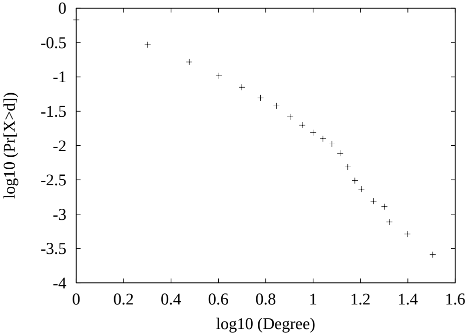

(a) Pansiot-Grad

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Linear growth, linear pref. attachment	$\gamma=3$	Barabási and Albert, 1999
Nonlinear preferential attachment $\Pi(k_i)\sim k_i^{\alpha}$	no scaling for $\alpha\neq 1$	Krapivsky, Redner, and Leyvraz, 2000
Asymptotically linear pref. attachment $\Pi(k_i)\sim a_{\infty}k_i$ as $k_i\to\infty$	$\gamma\to 2$ if $a_{\infty}\to\infty$ $\gamma\to\infty$ if $a_{\infty}\to 0$	Krapivsky, Redner, and Leyvraz, 2000
Initial attractiveness $\Pi(k_i)\sim A+k_i$	$\gamma=2$ if $A=0$ $\gamma\to\infty$ if $A\to\infty$	Dorogovtsev, Mendes, and Samukhin, 2000a, 2000b
Accelerating growth $\langle k\rangle\sim t^{\theta}$ constant initial attractiveness	$\gamma=1.5$ if $\theta\to 1$ $\gamma\to 2$ if $\theta\to 0$	Dorogovtsev and Mendes, 2001a
Accelerating growth $\langle k\rangle=at+2b$	$\gamma=1.5$ for $k\ll k_c(t)$ $\gamma=3$ for $k\gg k_c(t)$	Barabási *et al.*, 2001 Dorogovtsev and Mendes, 2001c
Internal edges with probab. p	$\gamma=2$ if $q=\frac{1-p+m}{1+2m}$	
Rewiring of edges with probab. q	$\gamma\to\infty$ if $p,q,m\to 0$	Albert and Barabási, 2000
c internal edges or removal of c edges	$\gamma\to 2$ if $c\to\infty$ $\gamma\to\infty$ if $c\to -1$	Dorogovtsev and Mendes, 2000c
Gradual aging $\Pi(k_i)\sim k_i(t-t_i)^{-\nu}$	$\gamma\to 2$ if $\nu\to-\infty$ $\gamma\to\infty$ if $\nu\to 1$	Dorogovtsev and Mendes, 2000b
Multiplicative node fitness $\Pi_i\sim\eta_i k_i$	$P(k)\sim\frac{k^{-1-C}}{\ln(k)}$	Bianconi and Barabási, 2001a
Additive-multiplicative fitness $\Pi_i\sim\eta_i(k_i-1)+\zeta_i$	$P(k)\sim\frac{k^{-1-m}}{\ln(k)}$ $1\leq m\leq 2$	Ergün and Rodgers, 2001
Edge inheritance	$P(k_{in})=\frac{d}{k_{in}^{\sqrt{2}}}\ln(ak_{in})$	Dorogovtsev, Mendes, and Samukhin, 2000c
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Redirection with probab. r	$\gamma=1+1/r$	Krapivsky and Redner, 2001
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Attaching to edges	$\gamma=3$	Dorogovtsev, Mendes, and Samukhin, 2001a
p directed internal edges $\Pi(k_i,k_j)\propto(k_i^{in}+\lambda)(k_j^{out}+\mu)$	$\gamma_{in}=2+p\lambda$ $\gamma_{out}=1+(1-p)^{-1}+\mu p/(1-p)$	Krapivsky, Rodgers, and Redner, 2001
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[Barabási, A.-L., and R. Albert, Statistical mechanics of complex networks, Reviews of Modern Physics, Vol 74, page 47-97, 2002.]

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Underlying geometry of vertices:

- A feature nodes have in many real-world networks.
- Often a reasonable hypothesis even when the nodes do not explicitly live in a metric space.

Central Question in this talk

How does underlying geometric structure affect preferential attachment?

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Old setup (Geo-PA-I):

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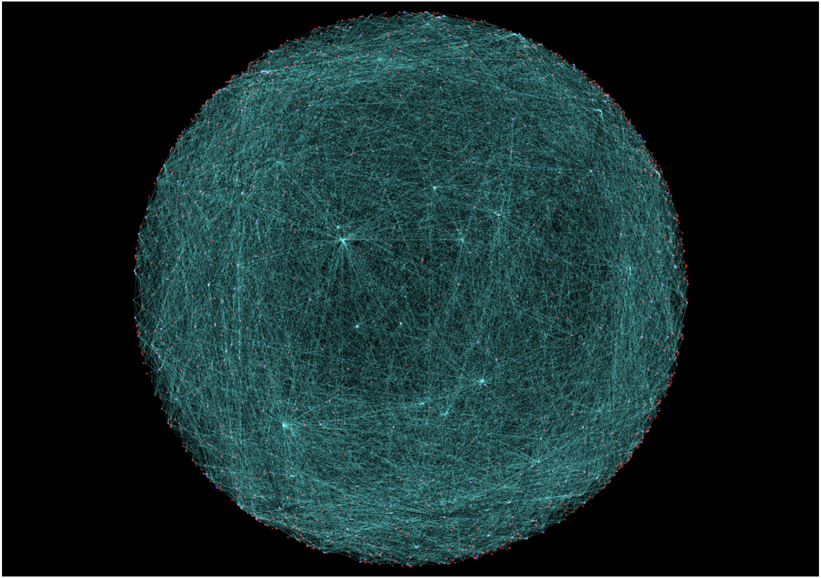
- We would like to take normalization Z to be

$$T_t(v_t) = \sum_{w:\|v_t - w\| \le r} \deg_t(w).$$

Geometric PA I Image



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Restrictions on F: I must exist, $0 < I < \infty$.

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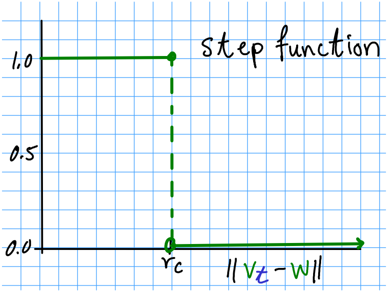

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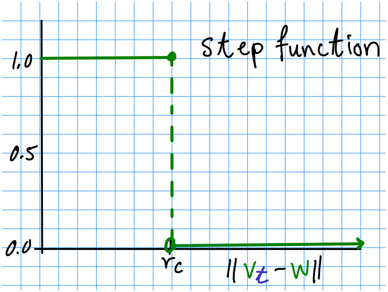

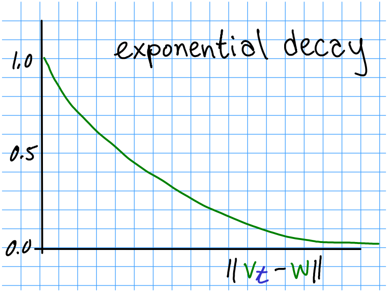

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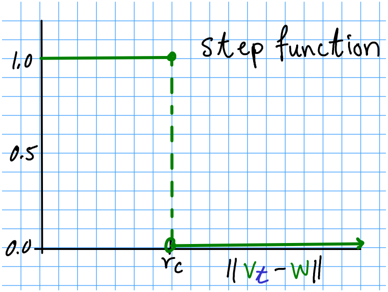

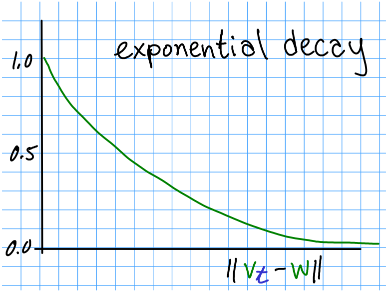

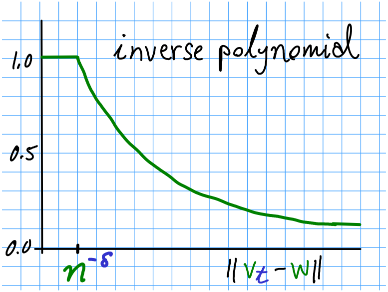

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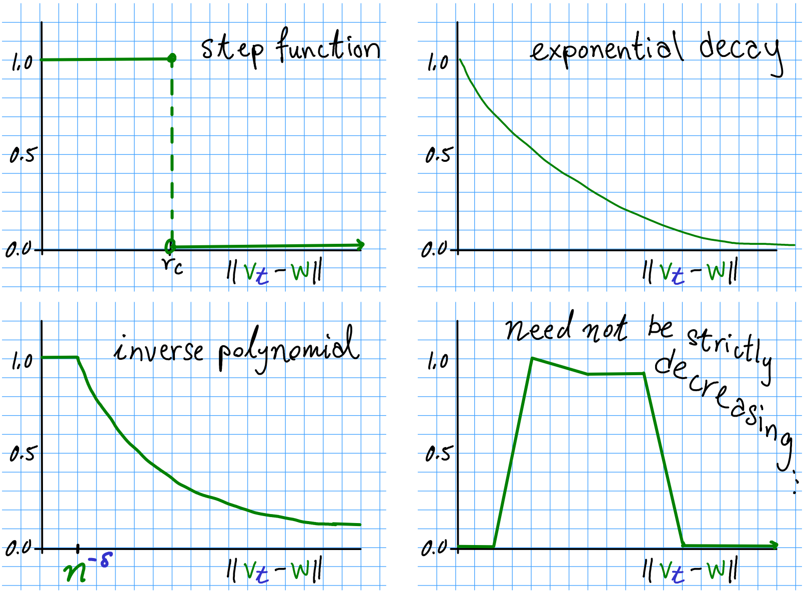

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- The diameter?

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(We also have a concentration result.)

Conductance/Sparsest cut

Theorem

For $\alpha > 0$ and m a sufficiently large constant, if there exist ϕ and η with

$$\frac{1}{n} \ll \phi \ll 1 \text{ and } \eta \ll 1$$

such that

$$\frac{1}{2} \int_{\eta}^{\pi} F(x) \sin x \, dx \leq \phi I$$

then the cut induced by a great circle of the sphere contains $\mathcal{O}((\eta + \phi)mn)$ edges **whp**.

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For $\beta < 2$, G is an expander.

Expander Criteria

Call F *tame* if exist constants C_1, C_2 such that

- $F(x) \geq C_1$ for $0 \leq x \leq \pi$,
- $I \leq C_2$.

Theorem

If $\alpha > 2$, F is tame, and $m \geq K \log n$ for sufficiently large K, then **whp**

- *G_n has conductance bounded below by a constant.*
- *G_n is connected.*
- *G_n has diameter $\mathcal{O}(\log n / \log m)$.*

Diameter

We also have some results for diameter when affinity function is not tame.

Lemma 1: a simple expectation

Lemma

For u chosen u.a.r. in S^2 and $t > 0$, we have

$$\mathrm{E}[T_t(u)] = 2lmt.$$

Proof

$$\begin{aligned}
\mathrm{E}[T_t(u)] &= \mathrm{E}\left[\sum_{w \in V_t} \deg_t(w) F(\|u - w\|)\right] \\
&= \sum_{w \in V_t} \deg_t(w) \int_{S^2} F(\|u - w\|) dw \\
&= \sum_{w \in V_t} \deg_t(w) I = 2Imt.
\end{aligned}$$

□

Lemma 2: a not-so-simple concentration inequality

Lemma

For any $t > 0$ *and for* u *chosen u.a.r. in* S^2*,*

$$\Pr\left[\left|T_t(u) - 2lmt\right| \geq ml(t^{2/\alpha} + t^{1/2}\ln t)\ln n\right] = \mathcal{O}\left(n^{-2}\right).$$

Proof by Azuma-Hoeffding, using a coupling argument.

Summary

Geo-PA-II: choose your own affinity function $F(x)$.

- Degree distribution has power $1 + \alpha$.
- Expander/Sparse cuts depend on $F(x)$.
- Diameter does as well.
- Proof uses tight concentration, coupling.

Future work

- Technical work:
  - $\alpha = 2$ (i.e. remove α)
  - non-uniform random points
  - necess. and suff. condition on F for expansion
- Modelling work: The sparse cuts are “wrong”.

Future work: getting sparse cuts right