Average-case analysis of some approximation algorithms for the metric uncapacitated facility location problem.

Abraham Flaxman, CMU
Alan Frieze, CMU
Juan Vera, CMU

Algorithmic goals in combinatorial optimization:

(1) In polynomial time,
(2) for every problem instance
(3) find an optimal solution.

Two out of three = research area:

Faster exponential time algorithms

(1) ~~In polynomial time~~,

(2) for every problem instance

(3) find an optimal solution.

Average-case analysis of algorithms

(1) In polynomial time,

(2) for ~~every~~ "most" problem instance

(3) find an optimal solution.

The most popular two out of three

Approximation algorithms

(1) In polynomial time,
(2) for every problem instance
(3) find ~~an optimal solution.~~ a solution within a known factor of optimal

This talk: Two two-out-of-threes

We will take

approximation algorithms

and see how good the apx ratio

is in the

average-case analysis

setting.

Facility location problem

Set $\mathcal{C}$ of cities

Set $\mathcal{F}$ of facilities

opening costs: f_i for facility i

connection costs: d_{ij} for city j to be served by facility i

Goal: Find set of facilities to open so that total cost is minimized.

Approx. fac. loc. state-of-the-art

- Connection costs unrestricted:

  $O(\log n)$ [Hochbaum]

- Connection costs form a metric

  1.52-apx exists [Mahdian, Ye, Zhang]

  1.46-apx does not [Guha, Khuller]

  (unless $NP \subseteq PTIME(n^{O(\log n)})$)

Brief description of an alg.

(1.86-apx; similar to & simpler than 1.52-apx, [Jain, Mahdian, Markakis, Saberi, Vazirani])

<u>Init</u>: Time $t = 0$. Funds $\delta_i = 0$ for each city.

Set of unconnected cities $\mathcal{U} = \mathcal{C}$

Brief description of an alg.

<u>Init</u>: Time $t=0$. Funds $\delta_i = 0$ for each city.
Set of unconnected cities $\mathcal{U} = \mathcal{C}$

<u>While $\mathcal{U} \neq \{\}$</u>:

For every $i \in \mathcal{U}$, increase δ_i simultaneously until: 1) $\delta_i = d_{ij}$ for some open facility j.
or 2) $\sum_{i \in \mathcal{U}} (\delta_i - d_{ij})^+ = f_j$ for some unopened facility j.

Take appropriate action.

Random instances (random how?)

Not $d_{i,j} \in [0,1]$ uniformly at random

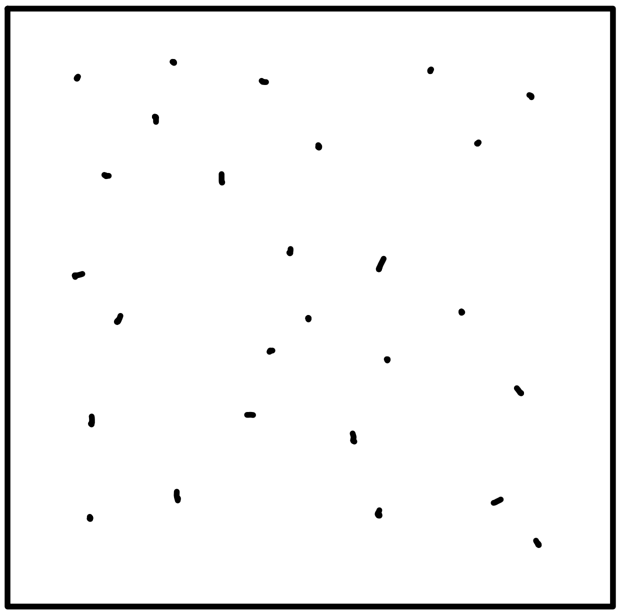
Geometric instances:

Place n cities u.a.r. in $[0,1]^2$.

Keep it simple:

- each city is a candidate facility
- opening costs all same value, f
- connection costs all given by ℓ_∞-norm

Random instances

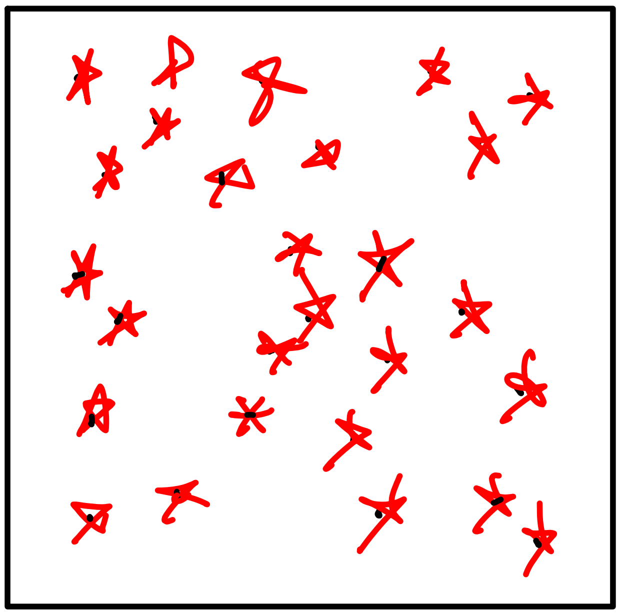


The opening cost f controls solution structure.

If $f = 0$ then every city opens as a fac.

If $f > n$ then just on city opens as a facility — the one most central.

Random instances

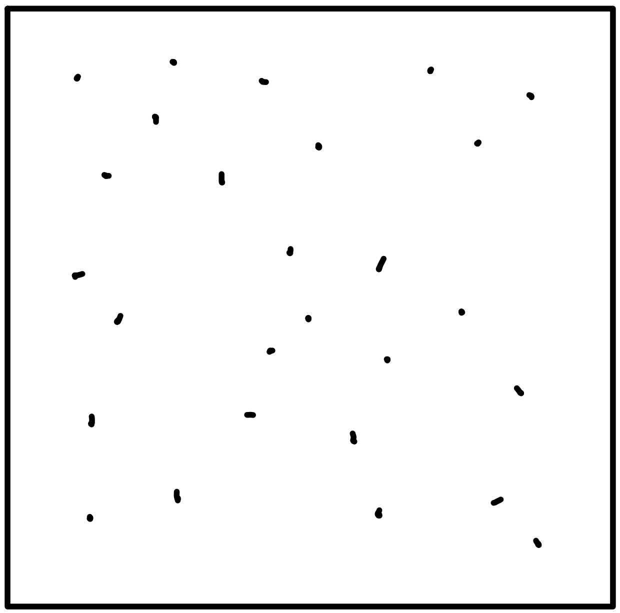


The opening cost f controls solution structure.

If $f = 0$ then every city opens as a fac.

If $f > n$ then just on city opens as a facility — the one most central.

Random instances

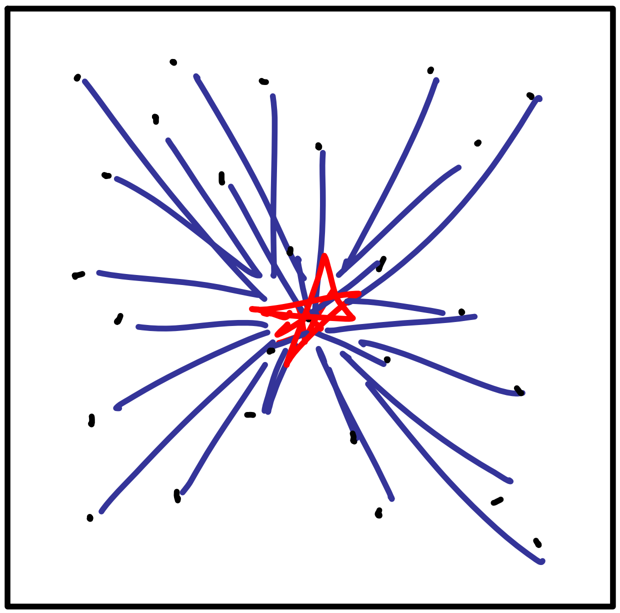


The opening cost f controls solution structure.

If $f = 0$ then every city opens as a fac.

If $f > n$ then just on city opens as a facility — the one most central.

Random instances

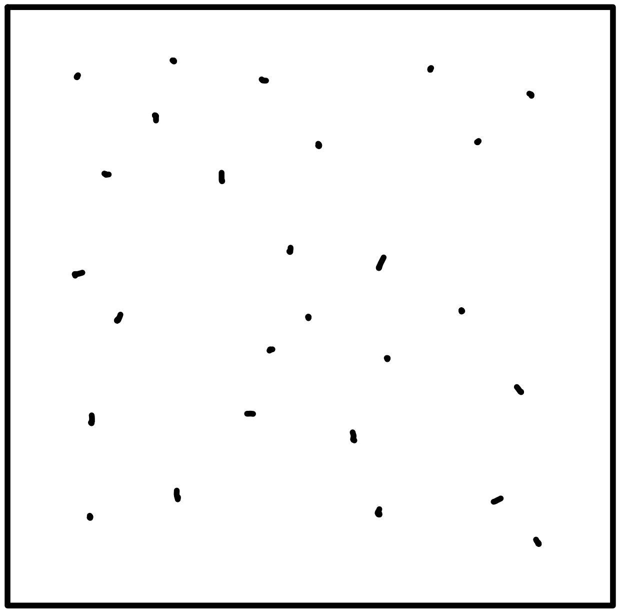


The opening cost f controls solution structure.

If $f = 0$ then every city opens as a fac.

If $f > n$ then just on city opens as a facility — the one most central.

Random instances



If $\frac{(\log n)^{3/2}}{n} \ll f \ll n$

then $OPT \sim \frac{1}{2}\alpha n$

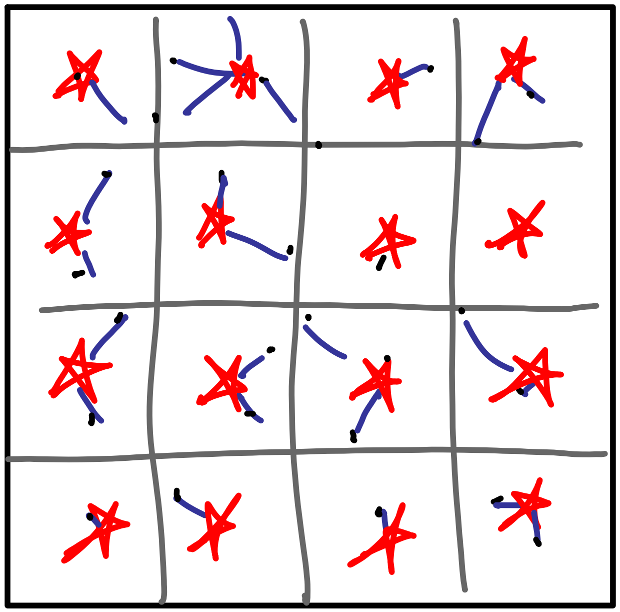
where $\alpha = \sqrt[3]{\frac{6f}{n}}$.

Random instances

If $\frac{(\log n)^{3/2}}{n} \ll f \ll n$

then $OPT \sim \frac{1}{2}\alpha n$

where $\alpha = \sqrt[3]{\frac{6f}{n}}$.



Pf: Partition square into $\alpha \times \alpha$ cells. Open facility near center of each cell. (Upper bound)

(Lower bound) Construct dual soln which is feasible whp.

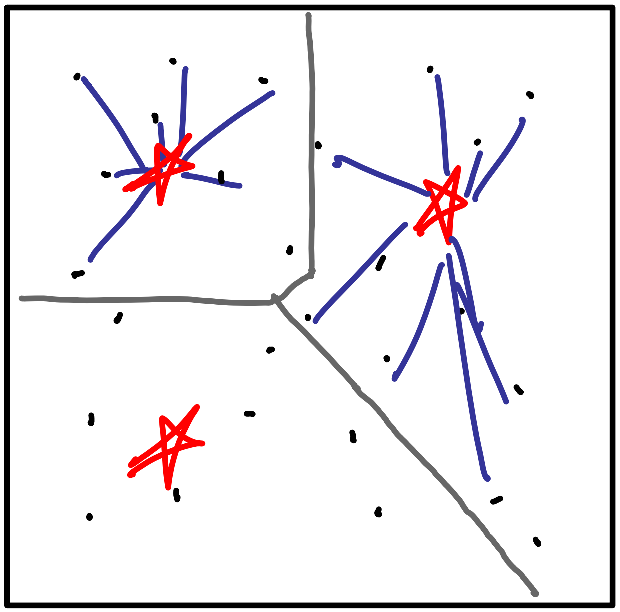
Main Theorem If $\frac{(\log n)^{3/2}}{n} \ll f \ll n$

then whp, $APX > (1+\varepsilon)OPT$

Pf: 1) Show that in any near-optimal soln, most of the Voronoi cells of open facilities must be close to square, close to centered, and have area approximately $\left(\frac{1}{\alpha}\right)^2$.

2) Show APX does not do this.

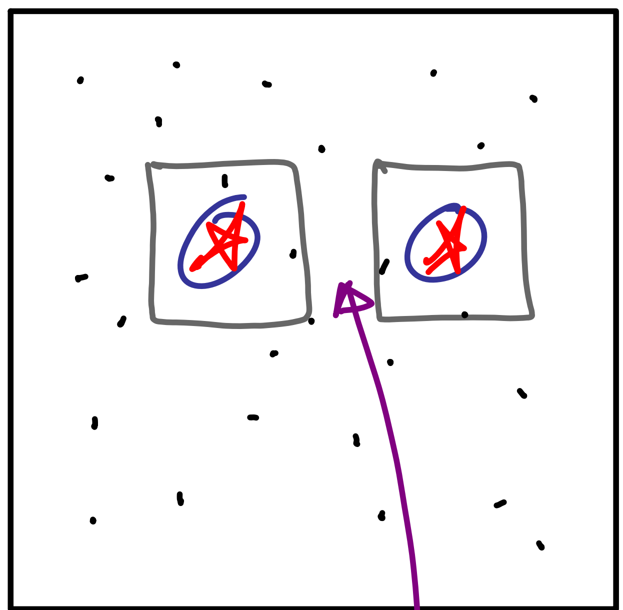
Proof of main theorem, part 1



A sort of isoperimetric inequality says the connection cost per facility is minimized when the Voronoi cell is square.

We quantize to a fine grid and prove a quantitative version of this.

Proof of main theorem, part 2



Each approx will open facilities at (dependent) random locations in the square.

Greedy can't fix Voronoi cells after this happens, which is the case often.

Summary

Whp, $APX > (1+\varepsilon)\,OPT$

Future work

- Tighter bounds on $\frac{APX}{OPT}$
  (finding it exactly is probably hard)

- See how other approx. algs do
  (for starters, analyze "Solve LP-Relax and round independently")